Anomalous energy transport in the FPU-β chain

Jani Lukkarinen*, Herbert Spohn†

Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, D-85747 Garching, Germany

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Abstract

We consider the energy current correlation function for the FPU-β lattice. For small non-linearity one can rely on kinetic theory. The issue reduces then to a spectral analysis of the linearized collision operator. We prove thereby that, on the basis of kinetic theory, the energy current correlations decay in time as $t^{-3/5}$. It follows that the thermal conductivity is anomalous, increasing as $N^{2/5}$ with the system size N.

1 Introduction and physical background

With the availability of the first electronic computing machines, Fermi, Pasta, and Ulam [1] investigated the dynamics of a chain of nonlinear oscillators, in particular, their relaxation to thermal equilibrium. Their work had a, in retrospect surprisingly, strong impact. We refer to the special issue [2] which accounts for the first fifty years. In our contribution, we will study the β-chain. This is a linear chain of equal mass particles which are coupled to their nearest neighbors by nonlinear springs with a potential of the form $U_\beta(r) = \frac{1}{8}r^2 + \frac{1}{4}\beta r^4$, with $\beta > 0$, and r being the string elongation. (According to the FPU convention, the α-chain has a nonlinearity $\frac{1}{3}\alpha r^3$ instead of $\frac{1}{4}\beta r^4$.) If we denote the particles positions by $q_i \in \mathbb{R}$, and their momenta (velocities) by $p_i \in \mathbb{R}$, then the β-chain has the Hamiltonian

$$H(q,p) = \sum_i \left[\tfrac{1}{2}p_i^2 + U_\beta(q_{i+1} - q_i)\right], \tag{1.1}$$

and the dynamics is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}q_i = p_i, \quad \frac{\mathrm{d}}{\mathrm{d}t}p_i = U_\beta'(q_{i+1} - q_i) - U_\beta'(q_i - q_{i-1}). \tag{1.2}$$

*E-mail: jlukkari@ma.tum.de
†E-mail: spohn@ma.tum.de

Over the last decade there has been a lot of interest to understand the energy transport through one-dimensional chains, amongst them the FPU β-chain [3]. Numerically, one common setup is to consider a chain of length N, and to couple its left- and rightmost particles to thermal reservoirs at temperatures T_- and T_+, respectively. For long times the chain relaxes to a steady state with a non-zero average energy current $j_{\mathrm{e}}(N) = (T_- - T_+)N^{-1}\kappa(N)$, and the interest lies in the dependence of $\kappa(N)$ on N for large N. For a regular transport, i.e., for transport satisfying Fourier's law, one has $\kappa(N) \to \mathrm{const.}$ for large N. Anomalous transport corresponds to $\kappa(N) \simeq N^\alpha$, with $0 < \alpha < 1$. In the β-chain more recent molecular dynamics simulations point to an α of approximately 0.4 [4, 5], and thus a larger energy transport than expected on the basis of Fourier's law. In these simulations chain lengths of up to $N = 2^{16}$ are used, and the result seems to be stable for a range of fairly low boundary temperatures. In [6] it is claimed that for somewhat higher boundary temperatures, there is a crossover at large N to $\kappa(N) \simeq N^{\frac{1}{3}}$. Hence, even on the numerical level the accurate value of α is still being debated.

In this paper, we will adopt a different, but physically equivalent procedure. One prepares initially (for $t = 0$) the infinite β-chain in thermal equilibrium at temperature $T > 0$. This means that the initial conditions of the Hamiltonian dynamics are distributed according to the (at this stage formal) Gibbs measure

$$Z^{-1}\mathrm{e}^{-H/T}\prod_{i\in\mathbb{Z}}[\mathrm{d}q_i\mathrm{d}p_i]\,. \tag{1.3}$$

This measure does not change in time. One now adds some extra energy close to the origin and studies the spreading of this excess energy. To be more precise, let us introduce the local energy, e_i, at the site $i \in \mathbb{Z}$ by

$$e_i(q,p) = \frac{1}{2}\left[p_i^2 + U_\beta(q_{i+1} - q_i) + U_\beta(q_i - q_{i-1})\right]. \tag{1.4}$$

We also employ the shorthand notation $e_i(t) = e_i(q(t), p(t))$, where $(q(t), p(t))$ is the solution to the Hamiltonian dynamics (1.2) for given initial conditions. Then we define the normalized local average excess energy by

$$S(i,t) = \frac{1}{\chi}\left(\langle e_i(t)e_0(0)\rangle - \langle e_i\rangle\langle e_0\rangle\right). \tag{1.5}$$

Here $\langle\cdot\rangle$ denotes the thermal average (1.3) over the initial conditions, and $\chi = \sum_i\left(\langle e_ie_0\rangle - \langle e_i\rangle\langle e_0\rangle\right)$ is a normalization guaranteeing $\sum_i S(i,0) = 1$. One has $S(-i,t) = S(i,t)$, and the energy spread at time t is defined as the spatial variance

$$D(t) = \sum_{i\in\mathbb{Z}} i^2 S(i,t). \tag{1.6}$$

Fourier's law corresponds to a diffusive spreading, $D(t) = \mathcal{O}(t)$ for large t, while an exponent $\alpha > 0$ corresponds to superdiffusive spreading with $D(t) =$

$\mathcal{O}(t^{1+\alpha})$. These properties can be more conveniently reformulated by introducing for each directed bond from i to $i+1$ a current $j_{i,i+1}$, so that the energy continuity equation holds in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} e_i + j_{i,i+1} - j_{i-1,i} = 0. \tag{1.7}$$

For the FPU-β model such a current observable is given by

$$j_{i,i+1}(q,p) = -\frac{1}{2}(p_{i+1} + p_i) U'_\beta(q_{i+1} - q_i). \tag{1.8}$$

Obviously, $\langle j_{i,i+1} \rangle = 0$. We next introduce the energy current-current correlation function

$$C_\beta(t) = \sum_{i \in \mathbb{Z}} \langle j_{0,1}(t) j_{i,i+1}(0) \rangle. \tag{1.9}$$

Then

$$D(t) = D(0) + \frac{1}{\chi} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' C_\beta(s - s'). \tag{1.10}$$

Note that $|C_\beta(t)| \le C_\beta(0) < \infty$. Clearly, if $\int_0^\infty \mathrm{d}t |C_\beta(t)| < \infty$, then $D(t) = \mathcal{O}(t)$. On the other hand, if $C_\beta(t) = \mathcal{O}(t^{\alpha-1})$ for large t with $0 < \alpha < 1$, then $D(t) = \mathcal{O}(t^{1+\alpha})$, and the spreading is superdiffusive.

The problem of regular versus anomalous energy transport may thus be rephrased as whether $C_\beta(t)$ decays integrably or not. Unfortunately, such a reformulation is of little help. To estimate the decay of a time correlation in equilibrium, such as $C_\beta(t)$, is an exceedingly difficult problem. However, in the limit of small β, through methods from kinetic theory, $C_\beta(t)$ may be expressed in a more accessible form. For the complete argument we refer to [7, 8, 9]. Here we only state the small β form of $C_\beta(t)$. To do so will require some preparation. But the goal of our contribution is to estimate the decay of $C_\beta(t)$ for the FPU-β chain in the limit of small β.

At $\beta = 0$, the system reduces to the harmonic Hamiltonian

$$H(q,p) = \sum_i \left[\tfrac{1}{2} p_i^2 + \tfrac{1}{8}(q_{i+1} - q_i)^2 \right], \tag{1.11}$$

which has the dispersion relation

$$\omega(k) = \sqrt{\tfrac{1}{2}(1 - \cos k)} = \left| \sin \frac{k}{2} \right|. \tag{1.12}$$

Here we use the convention that the discrete Fourier transform yields 2π-periodic functions, and also declare that the term "periodic function" always refers to a function which is 2π-periodic in all of its arguments. It will be convenient to choose

as the basic periodic cell the interval $I = [0, 2\pi)$. In particular, then $x \bmod 2\pi \in I$ for all $x \in \mathbb{R}$. On I the dispersion relation is simply

$$\omega(x) = \sin \frac{x}{2}, \tag{1.13}$$

and thus also for all $x \neq 0$,

$$\omega'(x) = \frac{1}{2} \cos \frac{x}{2}, \tag{1.14}$$

and we let arbitrarily $\omega'(0) = 0$. We also introduce

$$\Omega(x, y, z) = \omega(x) + \omega(y) - \omega(z) - \omega(x + y - z) \tag{1.15}$$

for $x, y, z \in \mathbb{R}$. With these conventions the linearized collision operator of the FPU-β lattice in the kinetic limit is given by

$$(Lf)(x) = \int_I \mathrm{d}y \int_I \mathrm{d}z \, \delta(\Omega(x, y, z)) \, (f(x) + f(y) - f(z) - f(x + y - z)), \tag{1.16}$$

with f periodically extended from I to $\mathbb{R}$, see [8].

L describes the collision of two phonons, where x, y label the incoming momenta and z, $x + y - z$ label the outgoing momenta, thus by fiat satisfying momentum conservation modulo 2π. Through the δ-function the collisions are also constrained to conserve energy. Note that at this stage, the definition in (1.16) is only formal since no prescription is given of how to deal with the δ-function. It turns out to be useful to consider $\tilde{L} = \omega L \omega$ as a linear operator on $L^2(I)$, with ω being understood as the multiplication operator by the function ω. We will prove later that $\tilde{L}$ is a bounded positive operator with a decomposition

$$\tilde{L} = W - A, \tag{1.17}$$

where A is compact and W is a multiplication operator.

Now we are in a position to state the conjectured behavior of $C_\beta(t)$ for small coupling β.

Kinetic conjecture: *For any $t \in \mathbb{R}$ and temperature $T > 0$*

$$\lim_{\beta \to 0^+} C_\beta(\beta^{-2} t) = \frac{T^2}{2\pi} \Big\langle \omega', \exp\Big[-\pi^{-1} (12T)^2 |t| \tilde{L} \Big] \, \omega' \Big\rangle, \tag{1.18}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(I)$. (A more detailed discussion about the scaling factors can be found in [8].)

Thus for small β, the decay of $C_\beta(t)$ is obtained from the spectral properties of $\tilde{L}$, certainly a more accessible item than the full Hamiltonian dynamics. Our goal here is to study the behavior of the kinetic correlation function

$$C(t) = \langle \omega', \mathrm{e}^{-|t| \tilde{L}} \omega' \rangle. \tag{1.19}$$

In kinetic theory, it is a common practice to use the relaxation time approximation, which in our case amounts to dropping the operator A, that is, to approximate

$$\langle \omega', \mathrm{e}^{-|t|\tilde{L}} \omega' \rangle \approx \langle \omega', \mathrm{e}^{-|t|W} \omega' \rangle. \tag{1.20}$$

As we will show, $W(x) = W(2\pi - x)$, and for $0 < x \ll 1$, $W(x)$ behaves asymptotically as $x^{5/3}$. Thus the relaxation time approximation predicts $\langle \omega', \mathrm{e}^{-|t|\tilde{L}} \omega' \rangle = \mathcal{O}(t^{-3/5})$ for large t, as has been derived in [10].

$\tilde{L}$ has the range of W as its essential spectrum. In particular, the essential spectrum starts from 0. Thus it is not obvious that the asymptotics predicted by the relaxation time approximation is really the correct one. To understand the time decay leads to two distinct mathematical issues.

(1) The so called collisional invariants, which in essence are zero modes of L, could in principle prevent $C(t)$ from decaying to 0. To exclude such a possibility, we have to characterize all collisional invariants, which involves solving a nontrivial functional equation.
(2) We will use the resolvent expansion to estimate $\langle \omega', \mathrm{e}^{-|t|\tilde{L}} \omega' \rangle$. In our case, it turns out to be necessary to make the expansion to the second order, yielding

$$\begin{aligned} \Big\langle \omega', \frac{1}{\lambda + \tilde{L}} \omega' \Big\rangle &= \Big\langle \omega', \frac{1}{\lambda + W} \omega' \Big\rangle + \Big\langle \omega', \frac{1}{\lambda + W} A \frac{1}{\lambda + W} \omega' \Big\rangle \\ &\quad + \Big\langle \omega', \frac{1}{\lambda + W} A \frac{1}{\lambda + \tilde{L}} A \frac{1}{\lambda + W} \omega' \Big\rangle. \end{aligned} \tag{1.21}$$

The first term is identical to the relaxation time approximation, and behaves as $\lambda^{-2/5}$ for $0 < \lambda \ll 1$. The second and third term will be shown to be $\mathcal{O}(\lambda^{-1/5-\varepsilon})$ for any $\varepsilon > 0$. Although also this second contribution is divergent, the first term is dominant, and thus we confirm the prediction of the relaxation time approximation in this particular case.

An inherent difficulty in resolvent expansions is the estimation of the remainder term, such as the last term in (1.21). Our method bears some similarity to the Birman-Schwinger estimates used in quantum mechanics. It relies on the fact that the resolvent expansion is made up to an even order, as well as on the operator $B = W^{-1/2} A W^{-1/2}$ being compact. In fact, it is likely that similar techniques can be used to study many of the cases where a decomposition $\tilde{L} = W - A$, with $W \geq 0$ and a compact B, is possible, although we would expect the optimal order for the resolvent expansion to vary from case to case. The exact order, as well as the exact power of the decay, would naturally depend also on the function ω'. A reader interested in such generalizations is invited to jump ahead to the proof of the main theorem in Section 6.

Our results imply that, on the kinetic time scale, the energy spread is superdiffusive, with $D(t) \simeq c\, t^{7/5}$, $c > 0$, for large t. This corresponds to a heat conduction exponent $\alpha = \frac{2}{5}$ and is in agreement with the molecular dynamics simulations of [4, 5]. As the example of long time tails in classical fluids teaches us, kinetic

theory might miss the true asymptotic decay of equilibrium correlation functions. Whether this is the case also for the FPU-β chain, remains a challenge for the future.

From the point of view of kinetic theory, our result is fairly surprising. Usually linearized collision operators have a spectral gap implying exponential decay of the current-current correlation function, and diffusive spreading for the corresponding conserved quantity. In fact, we are not aware of any other Boltzmann type kinetic model which would exhibit superdiffusive spreading.

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2 Main results

To define L, we first need to find all solutions to the energy constraint. The solution manifold to $\Omega(x,y,z)=0$, is clearly non-empty, as there are the *trivial solutions*

$$z = x \quad \text{and} \quad z = y. \tag{2.1}$$

We will later prove in Corollary 3.3 that, in addition, there is a solution $y = h(x,z)$, and that all other solutions are modulo 2π equal to one of these three. For $x,z \in I$ the function h is given by

$$h(x,z) = \frac{z-x}{2} + 2\arcsin\left(\tan\frac{|z-x|}{4}\cos\frac{x+z}{4}\right) \tag{2.2}$$

where $\arcsin$ denotes the principal branch with values in $[-\pi/2, \pi/2]$. We extend h to $\mathbb{R}^2$ by defining

$$h(x,z) = h(x \bmod 2\pi, z \bmod 2\pi) - i(x), \tag{2.3}$$

where $i(x) = x - (x \bmod 2\pi) \in 2\pi\mathbb{Z}$. This choice makes h everywhere continuous while ensuring that for all $x,z \in \mathbb{R}$, we still have $\Omega(x,h(x,z),z) = 0$.

The energy conservation δ-function can then be formally resolved by integrating over some chosen direction: for instance, choosing the y-integral for this purpose would yield for any $z \neq x$ and for any continuous periodic function G,

$$\begin{aligned}&\int_I \mathrm{d}y\, \delta(\Omega(x,y,z))G(x,y,z) \\ &\quad = \frac{1}{|\partial_2\Omega(x,z,z)|}G(x,z,z) + \frac{1}{|\partial_2\Omega(x,h(x,z),z)|}G(x,h(x,z),z).\end{aligned} \tag{2.4}$$

However, this procedure is somewhat suspect here, as it will lead to terms of the type $\infty - \infty$, related to the trivial solutions and canceled only due to symmetry properties. An additional difficulty lies in the application of the definition to functions G which are not continuous but merely L^2-integrable. To put the definition of L on a firmer ground, we will resort to a different approach in Section 3: we replace δ in (1.16) by a regularized δ-function $\delta_\epsilon(X) = \epsilon\pi^{-1}(\epsilon^2 + X^2)^{-1}$, $\epsilon > 0$, and then show that there is a unique self-adjoint operator L which agrees with these operators in the limit $\epsilon \to 0$. Our choice of regularization for the δ-function is not completely arbitrary: in the kinetic limit of lattice systems with random mass perturbations the corresponding δ-function also appears via a sequence of δ_ϵ-functions (see, for instance, Proposition A.1 in [11]).

A somewhat lengthy computation, to be discussed in Sections 3 and 4, shows that the formal procedure explained before is essentially correct: the trivial solutions give no contribution, and the unique limit operator L is

$$L = V + K_1 - 2K_2, \tag{2.5}$$

where K_1 and K_2 are integral operators determined by the integral kernels

$$K_1(x,y) = 4\frac{\mathbb{1}(F_-(x,y) > 0)}{\sqrt{F_-(x,y)}} \quad \text{and} \quad K_2(x,y) = \frac{2}{\sqrt{F_+(x,y)}}, \tag{2.6}$$

which are defined for $x, y \in I$ using the auxiliary functions

$$F_\pm(x,y) = \Big(\cos\frac{x}{2} + \cos\frac{y}{2}\Big)^2 \pm 4\sin\frac{x}{2}\sin\frac{y}{2}. \tag{2.7}$$

In addition, V denotes a multiplication operator by the function

$$V(x) = \int_I \mathrm{d}y\, K_2(x,y). \tag{2.8}$$

L was already used as the linearized collision operator in [10]. In addition to L, $\tilde{L} = \omega L \omega$, and $W = \omega^2 V$, the operator $B = W^{-1/2}(W - \tilde{L})W^{-1/2}$ will be of importance. Explicitly, B is then defined via the integral kernel

$$B(x,y) = V(x)^{-1/2}(2K_2(x,y) - K_1(x,y))V(y)^{-1/2}. \tag{2.9}$$

Let us next list the main properties of these operators, to be proven in Sections 4 and 5. We start with the results related to item 1 mentioned in the introduction.

Definition 2.1 *A measurable periodic function* $\psi : \mathbb{R} \to \mathbb{C}$ *is called a* collisional invariant *if for almost every* $x, y, z \in \mathbb{R}$ *such that* $\Omega(x,y,z) = 0$,

$$\psi(x) + \psi(y) - \psi(z) - \psi(x + y - z) = 0. \tag{2.10}$$

In the definition, "almost every" refers to the Lebesgue measure on any two-dimensional submanifold of the full solution set. The following theorem shows that, in the case considered here, there are only the obvious collisional invariants.

Theorem 2.2 *Suppose ψ is periodic and locally integrable: $\psi|_I \in L^1(I)$. Then ψ is a collisional invariant if and only if there are $c_1, c_2 \in \mathbb{C}$ such that $\psi(x) = c_1 + c_2\omega(x)$ for a.e. x.*

In higher dimensions there is a general argument which identifies the collisional invariants under minimal assumptions on ω [12]. In contrast, our proof here relies heavily on the specific form of ω, and does not exclude the appearance of non-trivial collisional invariants in some other one-dimensional systems.

Definition 2.3 *We define a* parity transformation *$P : L^2(I) \to L^2(I)$ by letting $(P\psi)(0) = \psi(0)$ and, for $x \in (0, 2\pi)$,*

$$(P\psi)(x) = \psi(2\pi - x). \tag{2.11}$$

Clearly, $\omega(x)$ is symmetric, and $\omega'(x)$ is antisymmetric under P.

Proposition 2.4 *$\tilde{L}$ is a bounded positive operator, and B is a compact self-adjoint operator on $L^2(I)$. Both B and $\tilde{L}$ commute with the parity operator P. In addition, $B \le 1$, and $B\psi = \psi$ if and only if the periodic extension of $V^{-1/2}\psi$ is a collisional invariant.*

Theorem 2.5 *Let $R : (0, \infty) \to \mathbb{R}_+$ be defined by*

$$R(\lambda) = \Big\langle \omega', \frac{1}{\lambda + \tilde{L}}\omega' \Big\rangle. \tag{2.12}$$

Then there is $0 < c_0 < \infty$ such that with $\alpha = \frac{2}{5}$

$$\lim_{\lambda \to 0^+} \lambda^\alpha R(\lambda) = c_0. \tag{2.13}$$

Since $R(\lambda) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda t} C(t)$, for $\lambda > 0$, $R(\lambda)$ is a Laplace transform of the monotonically decreasing positive function $C(t)$. Methods from Tauberian theory can then be used to connect the asymptotic behavior of R and C, proving that the asymptotic decay of the current-current correlations is given by $C(t) = \mathcal{O}(t^{-\frac{3}{5}})$, and that the integrated correlations grow like $\int_0^t \mathrm{d}s\, C(s) = \mathcal{O}(t^{\frac{2}{5}})$. Explicitly,

Corollary 2.6 *With $c_0 > 0$ and $\alpha = \frac{2}{5}$ as in Theorem 2.5, and with Γ denoting the gamma function, we have*

$$\lim_{t \to \infty} t^{1-\alpha} C(t) = \frac{c_0}{\Gamma(\alpha)}. \tag{2.14}$$

(For a proof of the result, see for instance “Zusatz zu Satz 2” on p. 208 of [13].)

We have divided the proof of the above results in four sections. We solve the energy constraint and derive the above form for the operator L in Section 3. Proposition 2.4 is proven in Section 4, which includes, in particular, the estimates proving the compactness of B. We study the collisional invariants in Section 5, and prove Theorem 2.2 there. Finally, these results are then applied in a resolvent expansion, and we prove Theorem 2.5 in Section 6. The short Appendix contains a convenient estimate for the norm of an integral operator.

3 Resolution of the energy constraint

We will define the operator L by the following procedure: we consider a regularization of the δ-function by

$$\delta_\epsilon(X) = \frac{\epsilon}{\pi}\frac{1}{\epsilon^2 + X^2}, \tag{3.1}$$

for any $0 < \epsilon \le 1$. Let L_ϵ denote the operator defined by (1.16) after δ has been replaced by δ_ϵ. This yields a bounded operator, for which using the symmetry properties of the integrand

$$4\langle f, L_\epsilon f\rangle = \int_{I^3} \mathrm{d}x\mathrm{d}y\mathrm{d}z\, \delta_\epsilon(\Omega(x,y,z))|f(x) + f(y) - f(z) - f(x+y-z)|^2. \tag{3.2}$$

Our aim in this section is to prove the following result about the limiting behavior of this quadratic form when $\epsilon \to 0^+$.

Proposition 3.1 *For any $f : \mathbb{R} \to \mathbb{C}$, which is periodic and Lipschitz continuous, the limit $\lim_{\epsilon\to 0^+}\langle f, L_\epsilon f\rangle$ exists, and it is non-negative, finite, and equal to*

$$\begin{aligned} &\int_{I^2} \mathrm{d}x\mathrm{d}z\, \frac{1}{2\sqrt{F_+(x,z)}}\,|f(x) + f(h(x,z)) - f(z) - f(x - z + h(x,z))|^2 \\ &\quad = \int_{I^2} \mathrm{d}x\mathrm{d}z\, f(x)^*(V(x) + K_1(x,z) - 2K_2(x,z))f(z). \end{aligned} \tag{3.3}$$

The proof of the Proposition will require some fairly technical estimates not needed later, and a reader accepting our definition of the operator L and the equality in (3.3) can well skip the proofs of the Lemmas below in the first reading.

We will begin by constructing the solutions to the energy constraint $\Omega = 0$, and then study the behavior of Ω around this set, in order to evaluate the limit of the approximate δ-functions. Let $D = [0, 2\pi]^3$ be the closure of I^3. For $(x,y,z) \in D$,

$$\Omega(x,y,z) = \sin\frac{x}{2} + \sin\frac{y}{2} - \sin\frac{z}{2} - \left|\sin\frac{x+y-z}{2}\right|. \tag{3.4}$$

Since then $-\pi \le \frac{x+y-z}{2} \le 2\pi$, we can split D into two sets U_+ and U_-, depending on the sign of the last term. Explicitly,

$$U_+ = \{(x,y,z) \in D \,|\, x + y - 2\pi \le z \le x + y\}, \tag{3.5}$$

$$U_- = \{(x,y,z) \in D \,|\, x + y \le z \text{ or } z \le x + y - 2\pi\}, \tag{3.6}$$

and $\Omega(x,y,z) = \Omega_\sigma(x,y,z)$ with $\sigma = +1$ if $(x,y,z) \in U_+$, and with $\sigma = -1$ if $(x,y,z) \in U_-$, where

$$\Omega_\sigma(x,y,z) = \sin\frac{x}{2} + \sin\frac{y}{2} - \sin\frac{z}{2} - \sigma\sin\frac{x+y-z}{2}. \tag{3.7}$$

The following representations of these functions will become useful later (they can be checked, for instance, by expressing the trigonometric functions in terms of complex exponentials): for all $x, y, z \in \mathbb{R}$,

$$\Omega_+(x,y,z) = 4 \sin\frac{x-z}{4} \sin\frac{y-z}{4} \sin\frac{x+y}{4}, \tag{3.8}$$

$$\Omega_-(x,y,z) = 2\left(\cos\frac{x+z}{4}\sin\frac{x-z}{4} + \cos\frac{x-z}{4}\sin\frac{2y+x-z}{4}\right). \tag{3.9}$$

From these, we directly find the zeroes of Ω:

Lemma 3.2 *Let $Z = \{(x,y,z) \in D \,|\, \Omega(x,y,z) = 0\}$, and denote $Z_\pm = Z \cap U_\pm$. Z_+ consists of those $(x,y,z) \in D$ for which either $z = x$, $z = y$, or $y = x \in \{0, 2\pi\}$. Z_- consists of those $(x,y,z) \in D$ which satisfy any of the following three conditions, where h is defined by (2.2) and (2.3),*

1. $x = 0$, $z = 2\pi$, *or* $x = 2\pi$, $z = 0$,
2. $x \le z$, *and* $y = h(x,z)$,
3. $x \ge z$, *and* $y = 2\pi + h(x,z)$.

In addition, for $(x,y,z) \in U_-$, with $x \ne z$, we have $\operatorname{sign}(z-x)\partial_y\Omega_-(x,y,z) \ge \cos^2\frac{x-z}{4}$.

Proof: By (3.8), $\Omega = 0$ on U_+ if and only if one of the three factors is zero. Since $|\frac{a-b}{4}| \le \frac{\pi}{2}$, and $0 \le \frac{a+b}{4} \le \pi$, for any $a, b \in \{x,y,z\}$, this can be checked to coincide with the above classification of Z_+.

To compute Z_-, let us first consider the case $\cos\frac{x-z}{4} = 0$. Then either $x = 0$, $z = 2\pi$, or $z = 0$, $x = 2\pi$, and both cases can be checked to form solutions for any y. Otherwise, $\cos\frac{x-z}{4} > 0$. Also $\cos\frac{x-z}{4} \ge \cos\frac{x+z}{4}$, as $|\frac{x-z}{4}| \le \frac{x+z}{4} \le \pi$. Similarly, as $|\frac{x-z}{4}| \le \frac{2\pi-x+2\pi-z}{4} \le \pi$, we have $\cos\frac{x-z}{4} \ge -\cos\frac{x+z}{4}$. Therefore, $|\cos\frac{x+z}{4}| \le \cos\frac{x-z}{4}$. Also by (3.9)

$$\partial_y\Omega_-(x,y,z) = \cos\frac{x-z}{4}\cos\frac{2y+x-z}{4}. \tag{3.10}$$

We then split the proof into three steps with additional conditions on x, z.

Assume first $z = x$. Then $(x,y,z) \in U_-$ if and only if $y = 0$ or $y = 2\pi$, and both cases clearly yield solutions. Since $h(x,x) = 0$, both cases are covered by the Lemma.

Assume then $z > x$. Then $(x,y,z) \in U_-$ if and only if $0 \le y \le z - x < 2\pi$. This implies that $|\frac{2y+x-z}{4}| \le \frac{z-x}{4} < \frac{\pi}{2}$, and thus in this case $\partial_y\Omega_- \ge \cos^2\frac{x-z}{4} > 0$. On the other hand, by explicit computation, then $\Omega_-(x,0,z) \le 0$ and $\Omega_-(x, z-x, z) \ge 0$. Therefore, for such x, z there is a *unique* solution $y \in [0, z-x]$. By (3.9) this solution satisfies

$$\sin\frac{2y+x-z}{4} = \cos\frac{x+z}{4}\tan\frac{z-x}{4}. \tag{3.11}$$

This equation has infinitely many solutions $y \in \mathbb{R}$, but the above bounds show that exactly one of them,

$$y = \frac{z-x}{2} + 2\arcsin\Big[\cos\frac{x+z}{4}\tan\frac{z-x}{4}\Big] = h(x,z), \tag{3.12}$$

with $\arcsin$ denoting the principal branch with values in $[-\pi/2, \pi/2]$, can belong to $[0, z-x]$. Since there must be a solution in this interval, we find that $h(x,z) \in [0, z-x]$, and thus $(x, h(x,z), z) \in U_-$.

To complete the analysis, assume $z < x$. Then $(x,y,z) \in U_-$ if and only if $2\pi + z - x \le y \le 2\pi$. We let $x' = 2\pi - x$, etc., when $z' > x'$, and $0 \le y' \le z' - x'$. As always $\Omega_-(x',y',z') = \Omega_-(x,y,z)$, we can conclude that for any x, z there is a unique solution in U_- which satisfies $y' = h(x',z')$, i.e., the solution is

$$y = 2\pi + \frac{z-x}{2} + 2\arcsin\Big[\cos\frac{x+z}{4}\tan\frac{x-z}{4}\Big] = 2\pi + h(x,z). \tag{3.13}$$

It also follows that in this case, $\partial_y \Omega(x,y,z) \le -\cos^2\frac{x-z}{4} < 0$ for all $2\pi + z - x \le y \le 2\pi$. This completes the proof of the Lemma. $\square$

As Ω is periodic, the Lemma yields immediately also a classification of the zeroes of Ω in $\mathbb{R}^3$.

Corollary 3.3 $\Omega(x,y,z) = 0$ *if and only if at least one of the following equalities holds modulo* 2π*:* $z = x$*,* $z = y$*, or* $y = h(x,z)$*.*

Proof: It is clear from the Lemma that any $(x,y,z) \in \mathbb{R}^3$ satisfying the above condition is a zero of Ω. For the converse, assume $\Omega(x,y,z) = 0$. Then for $x' = x \bmod 2\pi$, etc., also $\Omega(x',y',z') = 0$, and we can apply the Lemma. If $(x',y',z') \in U_+$, then either $z' = x'$, $z' = y'$ or $x' = y' = 0$. Since the last condition implies $z' = 0$, and thus $y' = 0 = h(0,0)$, also the last instance is covered in the Corollary. If the point belongs to U_-, we must have $y' = h(x',z') \bmod 2\pi$, and thus then $y = h(x,z)$ modulo 2π. $\square$

The following Lemma can then be used to compute the relevant limits for integrals over U_-:

Lemma 3.4 *Suppose* $G : \mathbb{R}^3 \to \mathbb{C}$ *is a periodic continuous function. Then*

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\int_{U_-} \mathrm{d}x\mathrm{d}y\mathrm{d}z\, \delta_\epsilon(\Omega(x,y,z))G(x,y,z)\\ &\quad = \int_{I^2} \mathrm{d}x\mathrm{d}z \frac{2}{\sqrt{F_+(x,z)}} G(x,h(x,z),z).\end{aligned} \tag{3.14}$$

Proof: As G is periodic and continuous, it is also bounded. Let $0 < \varepsilon < 1$ be arbitrary, and denote $X_\varepsilon = [0,\varepsilon] \times [2\pi-\varepsilon, 2\pi] \cup [2\pi-\varepsilon, 2\pi] \times [0,\varepsilon]$. Let us first consider some fixed $x, z \in [0,2\pi]^2 \setminus X_\varepsilon$, $x \ne z$. By Lemma 3.2, $|\partial_y \Omega| \ge$

$\cos^2 \frac{x-z}{4} \geq \sin^2 \frac{\varepsilon}{4} > 0$, and $y \mapsto \Omega(x, y, z)$ is a bijection with a unique zero, y_0, which is equal to $h(x, z)$ modulo 2π. We change the integration variable y to $t = \Omega(x, y, z)/\epsilon$, which shows that the integral over y is equal to

$$\int_{a/\epsilon}^{b/\epsilon} \mathrm{d}t \, \frac{1}{|\partial_2 \Omega(x, y(\epsilon t), z)|} \frac{1}{\pi(1+t^2)} G(x, y(\epsilon t), z). \tag{3.15}$$

with $a \leq 0$ and $b \geq 0$ and $y(0) = y_0$. This is always bounded by a constant which depends on ε but not on ϵ. Thus an application of the dominated convergence theorem shows that

$$\begin{aligned} \lim_{\epsilon \to 0^+} \int_{U_-} & \mathrm{d}x\mathrm{d}y\mathrm{d}z \mathbb{1}((x,z) \notin X_\varepsilon) \, \delta_\epsilon(\Omega(x, y, z)) G(x, y, z) \\ &= \int_{I^2} \mathrm{d}x\mathrm{d}z \frac{\mathbb{1}((x,z) \notin X_\varepsilon)}{|\partial_2 \Omega_-(x, h(x,z), z)|} G(x, h(x,z), z). \end{aligned} \tag{3.16}$$

We used here the observation that the set of $x = z$, as well as that of (x, z) for which $a = 0$ or $b = 0$, have zero measure. Here, by (3.10), we have

$$\begin{aligned} |\partial_2 \Omega_-(x, h(x,z), z)| &= \left| \cos \frac{x-z}{4} \right| \left(1 - \sin^2 \frac{2h + x - z}{4} \right)^{\frac{1}{2}} \\ &= \left(\cos^2 \frac{x-z}{4} - \sin^2 \frac{x-z}{4} \cos^2 \frac{x+z}{4} \right)^{\frac{1}{2}} = \frac{1}{2} \sqrt{F_+(x,z)}, \end{aligned} \tag{3.17}$$

where the last equality can be checked by a calculation, for instance, using the identity $\cos^2 u = \frac{1}{2}(1 + \cos(2u))$. For all $x, z \in I$, we clearly have an estimate

$$0 \leq K_2(x, z) = \frac{2}{\sqrt{F_+(x,z)}} \leq \left(\sin \frac{x}{2} \sin \frac{z}{2} \right)^{-1/2}. \tag{3.18}$$

Thus $F_+^{-1/2}$ is integrable, and we can again apply dominated convergence to take the limit $\varepsilon \to 0$ inside the integral. This proves that the right hand side of (3.16) converges to the right hand side of (3.14).

Therefore, to complete the proof of the Lemma, it is sufficient to prove that the integral over $(x, z) \in X_\varepsilon$ vanishes when first $\epsilon \to 0$ and then $\varepsilon \to 0$. In fact, using the symmetry between the two components of U_- and the boundedness of G, it is sufficient to study the integral

$$J_\varepsilon = \int_0^\varepsilon \mathrm{d}x \int_{2\pi - \varepsilon}^{2\pi} \mathrm{d}z \int_0^{z-x} \mathrm{d}y \, \frac{\epsilon}{\pi} \frac{1}{\epsilon^2 + \Omega_-^2}. \tag{3.19}$$

We split the integral over y into two parts at $y = \pi$. If $0 \leq y \leq \pi$, we have $2\partial_z \Omega_- \geq -\cos \frac{z}{2} \geq \cos \frac{\varepsilon}{2}$. Therefore, we can perform the z integral first, as above, and conclude that the result of the z-integral is uniformly bounded in ϵ and ε. Performing then the x and y integrals, shows that the full integral is bounded by

a constant times ε. In the remaining region, $\pi \leq y \leq z - x$, we have $2\partial_x\Omega_- \geq \cos\frac{x}{2} \geq \cos\frac{\varepsilon}{2}$. Thus in this case, we can perform the x integral first, with a uniformly bounded result. Performing then the y and z integrals, and combining the bound with the earlier estimate, proves that there is $c > 0$ such that $J_\varepsilon \leq c\varepsilon$. Thus $J_\varepsilon \to 0$ as $\varepsilon \to 0$, which concludes the proof of the Lemma. $\square$

To complete the proof of Proposition 3.1, we require one more Lemma, closely related to the above estimates.

Lemma 3.5 *Assume G is measurable and periodic on $\mathbb{R}^2$. Then*

$$\int_{I^2} \mathrm{d}x\mathrm{d}z \, \frac{1}{\sqrt{F_+(x,z)}} G(x, h(x,z)) = \int_{I^2} \mathrm{d}x\mathrm{d}y \, 2\frac{\mathbb{1}(F_-(x,y) > 0)}{\sqrt{F_-(x,y)}} G(x,y), \tag{3.20}$$

as long as either $G \geq 0$, or one of the above integrals is absolutely convergent.

Proof: As is apparent from (3.20), the proof is accomplished by a change of integration variables from z to $y = h(x,z)$ for a fixed x. However, even computing the local inverse functions from (2.2) does not appear to be completely straightforward. We will resort to a roundabout way, which relies on the fact that $h(x,z)$ is a solution to the energy constraint on Ω_-.

If $\tilde{h}(y;x)$ is a local inverse of $h(x,\cdot)$, then for all y in its domain there is $n \in \{0,1\}$ such that $(x, y + 2\pi n, \tilde{h}(y;x)) \in U_-$ and

$$\Omega(x, y, \tilde{h}(y;x)) = \Omega_-(x, h(x, \tilde{h}(y;x)), \tilde{h}(y;x)) = 0. \tag{3.21}$$

Conversely, assume that $x \in (0, 2\pi)$ is given, and $\tilde{h}(y;x)$ is a map from some interval $J \subset (0, 2\pi)$ to I such that either $x + y \leq \tilde{h}(y;x) \leq 2\pi$ or $0 \leq \tilde{h}(y;x) \leq x + y - 2\pi$ for all y, and $\Omega(x, y, \tilde{h}(y;x)) = 0$. Then by Lemma 3.2, we must have in the first case $y = h(x, \tilde{h}(y;x))$, and in the second case, $y = 2\pi + h(x, \tilde{h}(y;x))$ for all y.

Therefore, to construct all possible local inverse functions of $h(x,\cdot)$, it is sufficient to find for given x, y all z such that $(x,y,z) \in U_-$, and $\Omega_-(x,y,z) = 0$. We begin from the following representation of Ω_-: for all x, y, z,

$$\Omega_-(x,y,z) = 2\left(\sin\frac{x+y}{4}\cos\frac{x-y}{4} + \cos\frac{x+y}{4}\sin\frac{x+y-2z}{4}\right). \tag{3.22}$$

Let us assume that $(x,y,z) \in U_-$ with $0 < x < 2\pi$. Then $\cos\frac{x+y}{4} = 0$ implies $y = 2\pi - x$, and thus then $\Omega_- = 2\sin\frac{x}{2} > 0$. Thus if $\Omega_- = 0$, we have

$$\sin\frac{x+y-2z}{4} = -\tan\frac{x+y}{4}\cos\frac{x-y}{4}. \tag{3.23}$$

Since x, y, z are real, this is possible only if the absolute value of the right hand side is less than or equal to one. This condition is equivalent to the condition $F(x,y) \geq 0$, with

$$F(x,y) = \cos^2 \frac{x+y}{4} - \sin^2 \frac{x+y}{4} \cos^2 \frac{x-y}{4}. \tag{3.24}$$

A brief computation reveals that, in fact, $F(x,y) = \frac{1}{4} F_-(x,y)$, and thus $F_- \geq 0$ is a necessary condition to have any solutions.

When $F_-(x,y) \geq 0$, (3.23) holds if and only if there is $n \in \mathbb{Z}$ such that $z = z_n$, where

$$z_n = 2\pi n + \frac{x+y}{2} + (-1)^n 2 \arcsin\left(\tan \frac{x+y}{4} \cos \frac{x-y}{4}\right), \tag{3.25}$$

with $\arcsin$ denoting the principal branch. There can be maximally two values of n for which z_n belongs to $[0, 2\pi)$. However, by inspecting the sign of $\partial_3 \Omega_-$, similarly to what was done in the proof of Lemma 3.2, we find that for $F_- > 0$ there are exactly two solutions in U_-, and that for a given x there are maximally two values of y for which $F_- = 0$. If $F_- > 0$, the solutions are explicitly $z = \tilde{h}_\pm(y; x)$, where for either choice of the sign $\sigma \in \{\pm 1\}$

$$\tilde{h}_\sigma(y; x) = \frac{x+y}{2} + c_\sigma 2\pi + \sigma 2 \arcsin\left(\tan \frac{x+y}{4} \cos \frac{x-y}{4}\right), \tag{3.26}$$

and $c_\sigma = 0$, if $\sigma = +1$, and $c_\sigma = (-1)^{\mathbb{1}(x+y>2\pi)}$, if $\sigma = -1$.

Therefore, apart from a finite number of values $y \in [0, 2\pi]$, there are either no, or there are exactly two, solutions in U_-. Both of the solutions satisfy (3.23) and thus also for any such z

$$\begin{aligned} |\partial_3 \Omega(x,y,z)| &= \left|\cos \frac{x+y}{4}\right| \left(1 - \sin^2 \frac{x+y-2z}{4}\right)^{\frac{1}{2}} \\ &= \sqrt{F(x,y)} = \frac{1}{2}\sqrt{F_-(x,y)}. \end{aligned} \tag{3.27}$$

On the other hand, by implicit differentiation we find

$$\partial_2 \Omega_-(x, h(x,z), z) \partial_z h(x,z) + \partial_3 \Omega_-(x, h(x,z), z) = 0. \tag{3.28}$$

By (3.17) this implies

$$\frac{1}{2}\sqrt{F_+(x,z)}\, |\partial_z h(x,z)| = |\partial_3 \Omega_-(x, h(x,z), z)|, \tag{3.29}$$

which allows to compute the Jacobian of the change of variables.

Collecting all of the above results together, and applying Fubini's theorem, we can conclude that (3.20) holds for $G \geq 0$ and for any G which is bounded. This implies, in particular, that the integrals are equal if G is replaced by $|G|$ for any measurable G. Thus if either of these integrals in (3.20) is absolutely convergent, then the other must be so as well. Then an application of dominated convergence theorem proves that (3.20) holds also for such measurable G. □

Proof of Proposition 3.1: Let f be a periodic Lipschitz function. We express $\langle f, L_\epsilon f\rangle$ as an integral over I^3 using (3.2). As $D = U_+ \cup U_-$, and $D \setminus I^3$ and $U_+ \cap U_-$ have measure zero, we can split the integral into two parts by using $\int_{I^3} = \int_{U_+} + \int_{U_-}$. Since the factor multiplying δ_ϵ in the integrand is positive, periodic, and continuous, Lemma 3.4 implies that the integral over U_- converges to the left hand side of (3.3). By boundedness of f and applying (3.18), the integral yields a finite, non-negative result as claimed in the Proposition.

Thus in order to prove convergence to the left hand side of (3.3), we only need to show that the integral over U_+ vanishes as $\epsilon \to 0^+$. For any $(x,y,z) \in U_+$, using (3.8) and the fact that $|\sin x| \geq \frac{2}{\pi}|x|$ for $|x| \leq \frac{\pi}{2}$,

$$|\Omega_+(x,y,z)| \geq 4\frac{|x-z|}{2\pi}\frac{|y-z|}{2\pi}\sin\frac{x+y}{4} \geq \frac{1}{2\pi^3}|x-z||y-z|m(x,y), \tag{3.30}$$

where $m(x,y) = \min(x+y, 4\pi - x - y)$. This implies that, if $|x-z| \leq |y-z|$, then by the Lipschitz property of f, there is a constant $c > 0$ such that

$$\frac{1}{\epsilon^2 + \Omega_+^2}|f(x)+f(y)-f(z)-f(x+y-z)|^2 \leq \frac{c}{m(x,y)^2}\frac{|x-z|^2}{(\pi\epsilon)^2 + |x-z|^4}. \tag{3.31}$$

If $|y-z| \leq |x-z|$, the same estimate holds after x and y have been interchanged on the right hand side. Therefore,

$$\begin{aligned}&\int_{U_+} \mathrm{d}x\mathrm{d}y\mathrm{d}z\, \frac{\epsilon}{\pi}\frac{1}{\epsilon^2 + \Omega(x,y,z)^2}|f(x)+f(y)-f(z)-f(x+y-z)|^2 \\ &\quad \leq c\int_{I^2}\mathrm{d}x\mathrm{d}y\frac{\mathbb{1}(m(x,y)\geq\epsilon)}{m(x,y)^2}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\epsilon}{\pi}\frac{t^2}{(\pi\epsilon)^2+t^4} + c'\epsilon^2\epsilon^{1-2},\end{aligned} \tag{3.32}$$

where the second term is an estimate for the integral over (x,y) with $m(x,y) < \epsilon$ — these are contained in the two boxes $[0,\epsilon]^2$ and $[2\pi-\epsilon, 2\pi]^2$ and we have there estimated the integrand trivially using $\Omega^2 \geq 0$. The first integral over x, y is $\mathcal{O}(|\ln\epsilon|)$, and the second integral is $\mathcal{O}(\epsilon^{1/2})$, as seen by changing the integration variable to $s = \epsilon^{-1/2}t$. Therefore, we can conclude that the left hand side of (3.32) vanishes as $\epsilon \to 0^+$, i.e., that the integral over U_+ does not contribute to limit, as long as f is a Lipschitz function.

Thus to complete the proof of the Proposition, we only need to prove the equality in (3.3). Instead of doing this directly, let us come back to the integral over U_-, which was proven above to converge to the left hand side of (3.3). The set U_- is clearly invariant under $x \leftrightarrow y$. By inspection we check that this is also true for the map $z \mapsto z'$, with $z' = x+y-z+2\pi$, for $x+y \leq z$, and $z' = x+y-z-2\pi$, otherwise. Similarly, U_- is left invariant under the map $(x,y,z) \mapsto (x',y',z')$, with $y' = z$, $z' = y$, and $x' = x+y-z+2\pi$, for $x+y \leq z$, and $x' = x+y-z-2\pi$, otherwise. All of these maps leave also Ω invariant. We can thus first expand the

square and then use the above mappings to appropriately change variables to prove that

$$\frac{1}{4}\int_{U_-} \mathrm{d}x\mathrm{d}y\mathrm{d}z\, \delta_\epsilon(\Omega(x,y,z))|f(x)+f(y)-f(z)-f(x+y-z)|^2$$
$$= \int_{U_-} \mathrm{d}x\mathrm{d}y\mathrm{d}z\, \delta_\epsilon(\Omega(x,y,z)) f(x)^*(f(x)+f(y)-2f(z)). \qquad (3.33)$$

Lemma 3.4 can be applied to the right hand side proving that it converges to

$$\int_{I^2} \mathrm{d}x\mathrm{d}z\, \frac{2}{\sqrt{F_+(x,z)}} f(x)^*(f(x)+f(h(x,z))-2f(z)) \qquad (3.34)$$

By Fubini's theorem, the first and the last terms are equal to those of the right hand side of (3.3). The middle term is absolutely convergent by (3.18), and thus an application of Lemma 3.5 shows that it is equal to the missing K_1-term in (3.3). This completes the proof of the Proposition. □

4 Linearized collision operator

We will derive in this section the regularity properties of $\tilde{L}$ and B and prove Proposition 2.4. Let us recall the definition of L_ϵ in Section 3, and Proposition 3.1 proven there. Let also $\tilde{L}_\epsilon = \omega L_\epsilon \omega$. If f is a periodic Lipschitz function, then so is $g = \omega f$, and thus an immediate consequence of the Proposition is that then

$$\lim_{\epsilon\to 0^+} \langle f, \tilde{L}_\epsilon f\rangle = \lim_{\epsilon\to 0^+} \langle g, L_\epsilon g\rangle = \langle f, \tilde{L} f\rangle \geq 0. \qquad (4.1)$$

Here we have employed the definition of $\tilde{L}$ to identify it in the right hand side of (3.3). We shall soon prove that $\tilde{L}$ is a bounded operator on $L^2(I)$. As Lipschitz functions are dense in $L^2(I)$, the above result implies that $\tilde{L}$ is a positive operator. Moreover, $\tilde{L}$ is then *uniquely* determined by (4.1) in the following sense: Suppose L' is another self-adjoint operator (not necessarily bounded) for which $\langle f, L' f\rangle = \langle f, \tilde{L} f\rangle$ for every Lipschitz function f. Since then $L' - \tilde{L}$ is densely defined with $\langle f, (L'-\tilde{L})f\rangle = 0$, we can conclude using the polarization identity that $L'f = \tilde{L}f$ for all f Lipschitz. As $\tilde{L}$ is *bounded* and self-adjoint, this implies $L' = \tilde{L}$. Thus we only need to check that $\tilde{L}$ is bounded, and to show that $[P, \tilde{L}] = 0$, in order to conclude the properties stated about $\tilde{L}$ in Proposition 2.4. We remark in passing that only $\tilde{L}$ will be proven to be bounded, the operator L could well be unbounded.

For the proof of compactness of B, we need more precise estimates on W and on the kernel functions K_1 and K_2. We recall the definition of W given in Section 2, $W = \omega^2 V$. As to be shown, the exponent $\alpha = \frac{2}{5}$ in the main theorem is determined by the behavior of $W(x)$ near $x = 0$. This will be summarized in the following Lemma.

Lemma 4.1 *The function $W : \mathbb{R} \to \mathbb{R}_+$ is symmetric, $PW = W$, and continuous. In addition, there are constants $c_1, c_2 > 0$, such that for all $x \in \mathbb{R}$*

$$c_1 \left|\sin \frac{x}{2}\right|^{\frac{5}{3}} \le W(x) \le c_2 \left|\sin \frac{x}{2}\right|^{\frac{5}{3}} , \tag{4.2}$$

and also $\lim_{x\to 0} \left(|\sin \frac{x}{2}|^{-5/3} W(x)\right) = w_0 \in (0, \infty)$*, where*

$$w_0 = 4 \int_0^\infty \mathrm{d}s \, \left(2s + s^4\right)^{-\frac{1}{2}} . \tag{4.3}$$

Proof: If $x' = 2\pi - x$, then $\cos \frac{x'}{2} = -\cos \frac{x}{2}$, and $\sin \frac{x'}{2} = \sin \frac{x}{2}$. Therefore, $F_\pm(2\pi - x, 2\pi - y) = F_\pm(x, y)$, and a change of variables shows that $(PW)(x) = W(x)$ for all x. We will soon prove that the function

$$f(x) = \omega(x)^{-\frac{5}{3}} W(x) = \omega(x)^{\frac{1}{3}} \int_0^{2\pi} \mathrm{d}y \, \frac{2}{\sqrt{F_+(x, y)}}, \tag{4.4}$$

is continuous, with $f(0) = w_0 > 0$. This implies that f has a minimum and maximum on $[0, 2\pi]$, and since $f(x) > 0$, the minimum is non-zero. This will directly imply that W is continuous and satisfies the bounds in (4.2). Therefore, to complete the proof of the Lemma, we only need to study f.

Suppose $x \in (0, 2\pi)$. Then the bound (3.18) allows using the dominated convergence theorem to prove that $\lim_{h\to 0} f(x + h) = f(x)$, which proves that f is continuous at x. Thus we only need to prove that f is continuous at $x = 0$ and $x = 2\pi$ and, as $f(2\pi - x) = f(x)$ for all x, it suffices to study the limit $x \searrow 0$. Assume thus $0 < x < \pi/4$. Then for all $0 \le y \le \frac{3}{2}\pi$, we have $\cos(x/2) + \cos(y/2) \ge \cos(\pi/8) - \cos(\pi/4) > 0$, and thus also $F_+(x, y) \ge C > 0$. Therefore,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \int_0^{\frac{\pi}{2}} \mathrm{d}y \, \frac{2 s_x^{\frac{1}{3}}}{\sqrt{(c_x - c_y)^2 + 4 s_x s_y}}, \tag{4.5}$$

where $s_x = \sin(x/2)$, $c_x = \cos(x/2)$, etc. In the final integral, let us denote $\varepsilon = s_x$, and change variables to $s = \varepsilon^{-1/3} \sin \frac{y}{2}$. This shows that the integral is equal to

$$\int_0^{\varepsilon^{-\frac{1}{3}} 2^{-\frac{1}{2}}} \mathrm{d}s \, \frac{4\varepsilon^{\frac{1}{3}+\frac{1}{3}}}{\sqrt{1 - \varepsilon^{2/3} s^2}} \left[4\varepsilon^{1+\frac{1}{3}} s + \left(\frac{\varepsilon^{2/3} s^2 - \varepsilon^2}{\sqrt{1 - \varepsilon^{2/3} s^2} + \sqrt{1 - \varepsilon^2}} \right)^2 \right]^{-\frac{1}{2}} . \tag{4.6}$$

Here $\varepsilon^{2/3}$ can be canceled between the two factors. Now $x \to 0^+$ implies $\varepsilon \to 0^+$, and the limit can also be taken directly from the integrand, as a straightforward application of the dominated convergence theorem will show. Therefore, we can conclude (with a final change of variables to $s/2$) that

$$\lim_{x\to 0^+} f(x) = \int_0^\infty \mathrm{d}s \, \frac{4}{\sqrt{4s + \frac{1}{4} s^4}} = w_0. \tag{4.7}$$

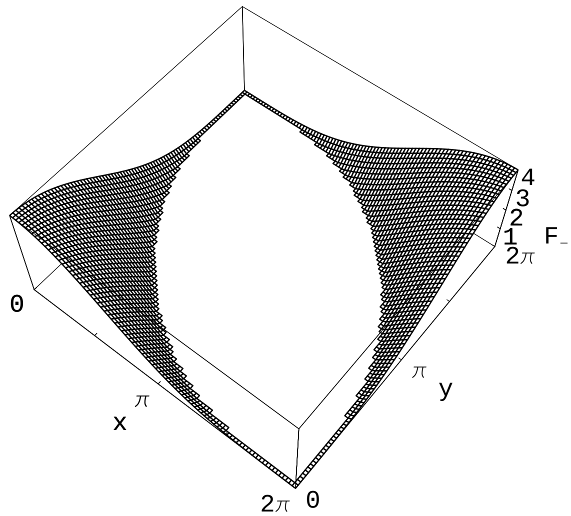

Figure 1: Plot of the positive part of $F_{-}(x, y)$.

Clearly, w_0 is strictly positive and finite, and thus by defining $f(2\pi n) = w_0$, f becomes a function which is continuous everywhere. This completes the proof of the Lemma. $\square$

We then require information about structure of singularities of the kernels K_1 and K_2 defined in (2.6). $K_2(x, z)$ is bounded apart from the point singularities at $(x, z) = (0, 2\pi)$ and $(2\pi, 0)$, and estimate (3.18) will suffice to control its behavior. In contrast, $K_1(x, y)$ has two line singularities of strength $\frac{1}{2}$, which coalesce at the corners $(x, y) = (0, 2\pi)$ and $(2\pi, 0)$ forming a point singularity of strength 1. To control these singularities, we will resort to the estimates given in the following Lemma. For the sake of illustration, we have plotted the positive part of F_{-} in Fig. 1.

Lemma 4.2 *Let $x \in (0, 2\pi)$ be given. Then there are y_1, y_2 such that $0 < y_1 < 2\pi - x < y_2 < 2\pi$, $F_{-}(x, y) \leq 0$ for $y_1 \leq y \leq y_2$, and*

$$F_{-}(x, y) \geq C(y_1 - y) \sin \frac{x}{2}, \quad \textit{for} \quad 0 \leq y < y_1, \tag{4.8}$$

$$F_{-}(x, y) \geq C(y - y_2) \sin \frac{x}{2}, \quad \textit{for} \quad y_2 < y \leq 2\pi, \tag{4.9}$$

with a constant $C > 0$ independent of x, y.

Proof: As $F_{-}(2\pi - x, 2\pi - y) = F_{-}(x, y)$, it suffices to prove the Lemma for $0 < x \leq \pi$. For notational simplicity, let $c = \cos \frac{x}{2}$ and $s = \sin \frac{x}{2}$. Then $0 < s \leq 1$ and $c = \sqrt{1 - s^2} \in [0, 1)$.

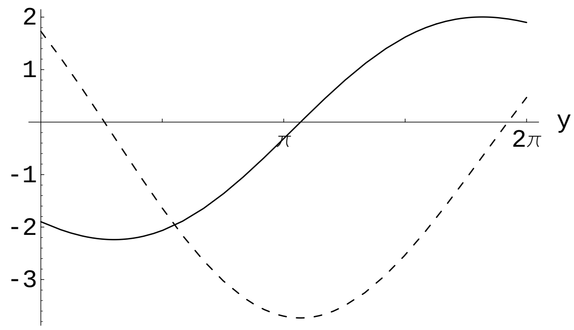

Figure 2: Plot of $\partial_2 F_-(x,y)$ (solid line) and of $F_-(x,y)$ (dashed line) for $x = 2.5$.

Since F_- is continuous and $F_-(x,0) = (1+c^2)^2 > 0$, $F_-(x,2\pi) = (1-c^2)^2 > 0$, and $F_-(x,2\pi - x) = -s^2 < 0$, we can find $0 < y_1 < 2\pi - x < y_2 < 2\pi$ such that $F_-(x,y_i) = 0$ and $F_-(x,y) \leq 0$ for $y_1 \leq y \leq y_2$. Assume then that $0 \leq y < y_1$, when $0 < x+y < 2\pi$. As $F_-(x,y_1) = 0$, we have

$$F_-(x,y) = -\int_y^{y_1} \mathrm{d}z\, \partial_2 F(x,z), \tag{4.10}$$

and to complete the proof of (4.8), it will be sufficient to show that $\partial_2 F_-(x,y) \leq -Cs$ for all $0 \leq y < y_1$. Similarly, to prove (4.9), it suffices to show that $\partial_2 F_-(x,y) \geq Cs$ for all $y_2 < y \leq 2\pi$.

Let us thus consider the function

$$F_2(y) = \partial_2 F_-(x,y) = -\sin\frac{y}{2}(c + \cos\frac{y}{2}) - 2s\cos\frac{y}{2}. \tag{4.11}$$

We claim that there are $y'_\pm$ such that $F_2(y)$ is strictly decreasing for $0 \leq y \leq y'_-$ and for $y'_+ \leq y \leq 2\pi$, and strictly increasing for $y'_- \leq y \leq y'_+$. As $F_2(0) = -2s < 0$ and $F_2(2\pi) = 2s > 0$, then there is a unique y'_1 such that $F_2(y'_1) = 0$. Then $y'_- < y'_1 < y'_+$, $F_2(y) < 0$ for $y < y'_1$, and $F_2(y) > 0$ for $y > y'_1$. Since F_- is then strictly decreasing up to $y = y'_1$ and after that strictly increasing, we can conclude that $y_1 < y'_1 < y_2$, and that $F_-(x,y) > 0$ for $y < y_1$ and for $y > y_2$. Therefore, to complete the proof of the Lemma, we only need to find $0 < C \leq 2$ such that $\partial_2 F_-(x,y_1) \leq -Cs$, and $\partial_2 F_-(x,y_2) \geq Cs$. To make the above argument more transparent, we have plotted a sample F_2 in Fig. 2.

Let us first consider estimating $\partial_2 F_-(x,y_2)$. Since $y_2 > 2\pi - x$, now $\cos\frac{y_2}{2} < -c$, and thus

$$\cos\frac{y_2}{2} + c = -2\sqrt{s\sin\frac{y_2}{2}}. \tag{4.12}$$

Therefore, denoting $t_2 = \sin\frac{y_2}{2}$,

$$F_2(y_2) = 2\sqrt{s}t_2^{3/2} + 2s\sqrt{1-t_2^2}. \tag{4.13}$$

Thus if $t_2^2 \le \frac{1}{2}$, $F_2(y_2) \ge \sqrt{2}s$. But also when $t_2^2 \ge \frac{1}{2}$, $F_2(y_2) \ge 2^{1-3/4}\sqrt{s} \ge 2^{1/4}s$. Therefore, always $\partial_2 F_-(x, y_2) \ge 2^{1/4}s$.

We estimate $\partial_2 F_-(x, y_1)$ next. Since $y_1 < 2\pi - x$, we have $\cos\frac{y_1}{2} > -c$, and

$$\cos\frac{y_1}{2} + c = 2\sqrt{s\sin\frac{y_1}{2}}. \tag{4.14}$$

Let $\varepsilon > 0$ be sufficiently small. If $y_1 \le \pi - 2\varepsilon$, then $\cos\frac{y_1}{2} \ge \sin\varepsilon > 0$, and thus $F_2(y_1) \le -2s\sin\varepsilon$. If $|y_1 - \pi| \le 2\varepsilon$, then $|\cos\frac{y_1}{2}| \le \sin\varepsilon$, and $\sin\frac{y_1}{2} \ge \cos\varepsilon$. Thus, by (4.14),

$$F_2(y_1) \le -2s\left[(\cos\varepsilon)^{\frac{3}{2}} - \sin\varepsilon\right]. \tag{4.15}$$

Therefore, choosing ε sufficiently small (for instance, $\varepsilon = \frac{1}{2}$) we have again obtained a bound of the required type.

We have thus proved the result for $y_1 \le \pi + 2\varepsilon$. Assume then $y_1 > \pi + 2\varepsilon$, and let $t_1 = \sin\frac{y_1}{2}$. Then $\cos\frac{y_1}{2} = -\sqrt{1-t_1^2}$, and (4.14) implies that

$$2\sqrt{st_1} = c - \sqrt{1-t_1^2} = \frac{t_1^2 - s^2}{c + \sqrt{1-t_1^2}} \le \frac{t_1^2}{2\sqrt{1-t_1^2}}. \tag{4.16}$$

Therefore, $t_1^{3/2} \ge 4\sqrt{1-t_1^2}\sqrt{s}$, and thus

$$F_2(y_1) = -2s^{\frac{1}{2}}t_1^{\frac{3}{2}} + 2s\sqrt{1-t_1^2} \le -2s\sqrt{1-t_1^2}(4-1) \le -6s\sin\varepsilon. \tag{4.17}$$

This proves that there is a pure constant $C > 0$ such that $\partial_2 F_-(x, y_1) \le -Cs$.

We still need to prove the monotonicity property of F_2 mentioned earlier. Let $u = \cos\frac{y}{2}$, when u goes from 1 to -1 strictly monotonicly, as y goes from 0 to 2π. Also

$$F_2(y) = -2su - (c+u)\sqrt{1-u^2} = g(u). \tag{4.18}$$

Then

$$g'(u) = -2s - \sqrt{1-u^2} + \frac{u(c+u)}{\sqrt{1-u^2}}, \quad g''(u) = \frac{c + 3u - 2u^3}{(1-u^2)^{3/2}}. \tag{4.19}$$

The polynomial $c + 3u - 2u^3$ has a local minimum at $u = -2^{-1/2}$ and a local maximum at $u = 2^{-1/2}$. Its values at $u = -1, 0, 1$ are $c-1, c, c+1$, respectively. Thus there is $-1 < u_0 \le 0$ such that $g''(u_0) = 0$, $g''(u) < 0$ for $u < u_0$ and $g''(u) > 0$ for $u > u_0$. Since $c < 1$, $g'(u)$ first decreases strictly from $+\infty$ to $g'(u_0)$ and then increases strictly to $+\infty$ again. Since $g'(0) < 0$, also $g'(u_0) < 0$ and thus there are $u_\pm$ such that g is strictly increasing for $-1 \le u \le u_-$ and for $u_+ \le u \le 1$ and strictly decreasing for $u_- \le u \le u_+$. This implies the stated monotonicity property of F_2 and completes the proof of the Lemma. $\square$

We then prove two intermediate compactness results which will become useful in the proof of Proposition 2.4. Instead of Sobolev-space techniques (such as proving that the operators improve a Sobolev index), we will rely on direct norm estimates which are quite straightforward in the present case.

Proposition 4.3 *Let $\psi : I \to \mathbb{C}$ be given, and assume that there are $C, p > 0$ such that*

$$|\psi(x)| \le C \left(\sin \frac{x}{2}\right)^p \tag{4.20}$$

for all $x \in I$. Then the function

$$K(x,y) = \psi(x)^* K_2(x,y)\psi(y) \tag{4.21}$$

defines a compact, self-adjoint integral operator on $L^2(I)$.

Proof: Since $\left(\sin \frac{x}{2}\right)^{p-1/2} \in L^2(I)$, the estimate (3.18) proves that K is Hilbert-Schmidt, and thus also compact. As $K(x,y)$ is symmetric, the operator is self-adjoint. $\square$

Proposition 4.4 *Let $\psi : I \to \mathbb{C}$ satisfy the assumptions of Proposition 4.3. Then the function*

$$K(x,y) = \psi(x)^* K_1(x,y)\psi(y) \tag{4.22}$$

defines a compact, self-adjoint integral operator on $L^2(I)$.

Proof: Let $p, C > 0$ be constants for which (4.20) holds. As the bound is a decreasing function of p, it is sufficient to prove the Proposition assuming $0 < p \le \frac{1}{2}$. Suppose $0 < \varepsilon < 2\pi$ is arbitrary, and let T_ε denote the integral operator defined by

$$X_\varepsilon = \left\{ (x,y) \in [\varepsilon, 2\pi - \varepsilon]^2 \,\middle|\, F_-(x,y) \ge C\varepsilon \sin \frac{\varepsilon}{2} \right\}, \tag{4.23}$$

$$T_\varepsilon(x,y) = \mathbb{1}((x,y) \in X_\varepsilon) K(x,y). \tag{4.24}$$

Then $|T_\varepsilon(x,y)| \le c(\varepsilon \sin \frac{\varepsilon}{2})^{-1/2}$ for some constant c, and since the kernel $T_\varepsilon(x,y)$ is obviously symmetric, we can conclude that T_ε is a self-adjoint Hilbert-Schmidt operator on $L^2(I)$. As $K(x,y)$ is also symmetric, we can apply Proposition A.1 to the integral operator $K - T_\varepsilon$. We will choose $\phi(x) = \sin(x/2)^p$, $\alpha = 1 - \frac{1}{2p}$, and $A(x,y) = \sin(x/2)^{-p}(K(x,y) - T_\varepsilon(x,y))\sin(y/2)^{-p}$. By (4.20), then $|A(x,y)| \le C^2 \mathbb{1}((x,y) \notin X_\varepsilon) K_1(x,y)$. Therefore, it is enough to inspect the integral

$$J(x) = \int_I \mathrm{d}y \, \left(\sin \frac{x}{2}\right)^{p+\frac{1}{2}} \left(\sin \frac{y}{2}\right)^{p-\frac{1}{2}} \mathbb{1}((x,y) \notin X_\varepsilon) K_1(x,y). \tag{4.25}$$

We claim that there is $c > 0$, such that $J(x) \le c\varepsilon^p$ for all $x \in I$. Then by Proposition A.1, we have $\|K - T_\varepsilon\| \le C^2 c\varepsilon^p$. Since this also holds for $\varepsilon > \pi$, when $T_\varepsilon = 0$, we find that K is itself a self-adjoint bounded operator. However, then also $T_\varepsilon \to K$ in norm, and since each T_ε is a compact operator, and the space of compact operators is closed in the operator norm, we conclude that K is also compact, proving the results mentioned in the theorem.

Thus we only need to show that $J(x) \le c\varepsilon^p$ for all $x \in I$, assuming $0 < p \le \frac{1}{2}$. For any $x \in I$ we obtain from Lemma 4.2 the following rough estimate, where the integration region is estimated trivially,

$$J(x) \le C^{-1/2}\Big(\sin\frac{x}{2}\Big)^p \Big(\int_0^{y_1} \mathrm{d}y\, f_1(y) + \int_{y_2}^{2\pi} \mathrm{d}y\, f_2(y)\Big), \tag{4.26}$$

where

$$f_1(y) = \Big(\sin\frac{y}{2}\Big)^{p-\frac{1}{2}} \frac{1}{\sqrt{y_1 - y}} \quad \text{and} \quad f_2(y) = \Big(\sin\frac{y}{2}\Big)^{p-\frac{1}{2}} \frac{1}{\sqrt{y - y_2}}. \tag{4.27}$$

Since

$$\int_0^{y_1} \mathrm{d}y\, y^{p-\frac{1}{2}}(y_1 - y)^{-\frac{1}{2}} = y_1^p \int_0^1 \mathrm{d}t\, t^{p-\frac{1}{2}}(1-t)^{-\frac{1}{2}}, \tag{4.28}$$

both of the remaining integrals are uniformly bounded in x, independently of the actual values of y_1 and y_2. The rough estimate proves that $\sup_x J(x)$ is uniformly bounded for all ε, but it also proves that if $x \in [0, \varepsilon)$ or if $x \in (2\pi - \varepsilon, 2\pi)$, then there is a constant c' such that $J(x) \le c'\varepsilon^p$.

Let us then consider the remaining case when ε is small (say $\varepsilon < 1$) and $x \in [\varepsilon, 2\pi - \varepsilon]$. Then $\sin\frac{x}{2} \ge \sin\frac{\varepsilon}{2}$, and thus Lemma 4.2 shows that every $(x, y) \in I^2$ for which $\varepsilon \le y \le y_1(x) - \varepsilon$ or $y_2(x) + \varepsilon \le y \le 2\pi - \varepsilon$, belongs to X_ε. Therefore, such y do not contribute to $J(x)$, and we can estimate

$$\begin{aligned} J(x) \le C^{-1/2}\Big(\sin\frac{x}{2}\Big)^p \Big(&\int_0^{\min(\varepsilon, y_1)} \mathrm{d}y\, f_1(y) + \int_{\max(0, y_1-\varepsilon)}^{y_1} \mathrm{d}y\, f_1(y) \\ &+ \int_{y_2}^{\min(2\pi, y_2+\varepsilon)} \mathrm{d}y\, f_2(y) + \int_{\max(2\pi-\varepsilon, y_2)}^{2\pi} \mathrm{d}y\, f_2(y)\Big). \end{aligned} \tag{4.29}$$

Each of the four integrals can be estimated similarly to (4.28), which shows that they are bounded by $c''\varepsilon^p$ for some constant c''. Therefore, we have proven that also in this case $J(x) \le c\varepsilon^p$ for some c. This completes the proof of the Proposition. $\square$

Proof of Proposition 2.4: By Propositions 4.3 and 4.4, the operators $\omega K_1 \omega$ and $\omega K_2 \omega$ are compact, and thus also bounded. By Lemma 4.1, W is a bounded function, and thus $\tilde{L}$ is a sum of a bounded multiplication operator and a compact integral operator. Thus $\tilde{L}$ is bounded, and then the argument in the beginning of the section, based on Proposition 3.1, implies that it is positive. On the other hand,

Lemma 4.1 also implies that $0 \leq V(x)^{-1/2} = \omega(x)W(x)^{-1/2} \leq c_2 \sin(x/2)^{1/6}$, and we can apply Propositions 4.3 and 4.4 also to the definition of B, equation (2.9). This proves that B is a compact, self-adjoint operator on $L^2(I)$. B and $\tilde{L}$ also commute with P, by the symmetry properties of $F_\pm$ stated in the beginning of the proof of Lemma 4.1.

Thus we only need to prove the last claim in the Proposition. Applying the definitions, we find $\tilde{L} = W^{1/2}(1-B)W^{1/2}$. For any $\psi \in L^2$, for which $W^{-1/2}\psi \in L^2(I)$, we can find a sequence f_n of Lipschitz continuous functions, such that $f_n \to W^{-1/2}\psi$ in L^2. Thus by the boundedness of $\tilde{L}$ and Proposition 3.1, then

$$\langle\psi,(1-B)\psi\rangle = \lim_n \langle f_n, \tilde{L} f_n\rangle = \lim_n \int_{I^2} \mathrm{d}x\mathrm{d}z \frac{1}{2\sqrt{F_+(x,z)}} \times |g_n(x) + g_n(h(x,z)) - g_n(z) - g_n(x-z+h(x,z))|^2 \,, \tag{4.30}$$

where $g_n = \omega f_n \to V^{-1/2}\psi$. The function defined by the integral on the right hand side is L^2-continuous in f_n. To see this, let us inspect the difference of two such integrals, which can be bounded by a sum of finitely many terms of the type $\int \mathrm{d}x\mathrm{d}z(4F_+)^{-1/2}|G(X)|^2$, where X denotes any one of the functions x, z, $h(x,z)$, or $x-z+h(x,z)$, and G is in L^2. The first two choices of X lead to integrals which clearly can be bounded by $\int \mathrm{d}x V(x)|G(x)|^2$. However, so do the last two choices, as can be seen by employing the symmetry $h(z,x) = x-z+h(x,z)$ and Lemma 3.5. Using then the fact that any relevant G is of the form $G = \omega F$, $F \in L^2$, we have here $\int \mathrm{d}x V(x)|G(x)|^2 = \int \mathrm{d}x W(x)|F(x)|^2 \leq \|W\|_\infty \|F\|^2$. This suffices to prove the continuity, and thus for the above class of ψ,

$$\langle\psi,(1-B)\psi\rangle = \int_{I^2} \mathrm{d}x\mathrm{d}z \frac{1}{2\sqrt{F_+(x,z)}} \times |g(x) + g(h(x,z)) - g(z) - g(x-z+h(x,z))|^2 \,, \tag{4.31}$$

with $g = V^{-1/2}\psi$. Then the previous argument can also be applied to show that the right hand side is L^2-continuous in ψ, which proves that (4.31) holds for all $\psi \in L^2$. Therefore, $1-B \geq 0$, and $B\psi = \psi$ if and only if the integral on the right hand side of (4.31) vanishes for $g = V^{-1/2}\psi$. Since then the integrand must be zero almost everywhere, this is possible if and only if the periodic extension of g is a collisional invariant. □

5 Collisional invariants (proof of Theorem 2.2)

It is clear that every $\psi(x) = c_1 + c_2\omega(x)$ is a locally integrable collisional invariant. Thus to prove the Theorem, it will be enough to consider any ψ, which is a locally integrable collisional invariant, and to show that it is almost everywhere equal to a function of the above form. Let us assume ψ is such a function. Then, as $\Omega(x, h(x,z), z) = 0$ for all x, z, we have for almost every $x, z \in \mathbb{R}$

$$\psi(x) + \psi(h(x,z)) - \psi(z) - \psi(x-z+h(x,z)) = 0. \tag{5.1}$$

In addition, since $h(2\pi - x, 2\pi - z) = -h(x,z)$ for $x, z \in I$, then also $P\psi$ satisfies (5.1) almost everywhere. Therefore, both of $(\psi \pm P\psi)/2$ have this property, and thus it is sufficient to prove the result assuming that ψ is either symmetric of antisymmetric under P.

Let us begin by showing that then there is $f : \mathbb{R} \to \mathbb{C}$ which is periodic and twice continuously differentiable apart possibly from points in $2\pi\mathbb{Z}$, and for which $\psi = f$ almost everywhere. We will do this by integrating (5.1) over x. However, the integration region has to be chosen with some care, in order to guarantee that the result is finite. With a certain $0 < \varepsilon_0 < \frac{\pi}{4}$ to be fixed later, we consider an arbitrary $0 < \varepsilon \le \varepsilon_0$. We define for all $\varepsilon \le z \le \pi + \varepsilon$

$$f_{1,\varepsilon}(z) = \frac{1}{\varepsilon} \int_{2\pi-\varepsilon}^{2\pi} \mathrm{d}x \, [\psi(x) + \psi(h(x,z)) - \psi(x - z + h(x,z))] \,. \tag{5.2}$$

A comparison with (5.1) reveals that then $f_{1,\varepsilon}(z) = \psi(z)$ almost everywhere. On the whole integration region $x > z$ and thus

$$h(x,z) = \frac{z-x}{2} + \Phi(x,z) \quad \text{and} \quad x - z + h(x,z) = \frac{x-z}{2} + \Phi(x,z) \tag{5.3}$$

with

$$\Phi(x,z) = 2 \arcsin\left(\tan \frac{x-z}{4} \cos \frac{x+z}{4} \right) . \tag{5.4}$$

We claim that there are $\varepsilon_0, C > 0$ such that for all x, z, ε as above

$$2\partial_x \Phi(x,z) \le -1 - C. \tag{5.5}$$

Together with (5.3) this implies that the last two mappings in the arguments of ψ in (5.2) are strictly decreasing in x. In particular, when $x \to 2\pi$, we have also $h(x,z) \searrow z - 2\pi$ and $x - z + h(x,z) \searrow 0$. Thus a change of variables and denoting $\zeta_0 = h(2\pi - \varepsilon, z)$ yields

$$\begin{aligned} f_{1,\varepsilon}(z) = \frac{1}{\varepsilon} \int_{2\pi-\varepsilon}^{2\pi} \mathrm{d}x \, \psi(x) + \frac{1}{\varepsilon} \int_{z-2\pi}^{\zeta_0} \mathrm{d}\zeta \, \psi(\zeta) \frac{2}{1 - 2\partial_x \Phi(x_1(\zeta,z),z)} \\ - \frac{1}{\varepsilon} \int_0^{2\pi-\varepsilon-z+\zeta_0} \mathrm{d}\zeta \, \psi(\zeta) \frac{2}{-1 - 2\partial_x \Phi(x_2(\zeta,z),z)} . \end{aligned} \tag{5.6}$$

Since both of the factors multiplying $\psi(\zeta)$ are continuous in z and uniformly bounded, we can conclude using the dominated convergence theorem that $f_{1,\varepsilon}$ is continuous.

As mentioned before, $\psi(z) = f_{1,\varepsilon}(z)$ for almost every $\varepsilon \le z \le \pi + \varepsilon$. Since $P\psi = \sigma\psi$, with $\sigma \in \{\pm 1\}$, we can define $f_\varepsilon(z) = f_{1,\varepsilon}(z)$ for $\varepsilon \le z \le \pi + \varepsilon$, and $f_\varepsilon(z) = \sigma f_{1,\varepsilon}(2\pi - z)$ for $\pi - \varepsilon \le z \le 2\pi - \varepsilon$. In the common domain near $z = \pi$ both functions have to be equal *everywhere*, as they are continuous and coincide with ψ almost everywhere. In particular, $f_\varepsilon(z)$ is also everywhere

continuous and equal to ψ almost everywhere. Since ε was arbitrary, we can then extend the definition to cover the whole of $(0, 2\pi)$, by choosing $f(z) = f_\varepsilon(z)$ for any $\varepsilon < z, 2\pi - z$. Again, by continuity, any two functions f_ε and $f_{\varepsilon'}$ must agree on the intersection of their domains of definition, so f is a continuous function on $(0, 2\pi)$, which we extend periodically to $\mathbb{R}$. Then $\psi(z) = f(z)$ a.e. $z \in \mathbb{R}$.

Using the continuity of h, this implies that

$$f(x) + f(h(x, z)) - f(z) - f(x - z + h(x, z)) = 0, \tag{5.7}$$

for all $x, z \in I$ for which all arguments are non-zero, i.e., whenever $x \neq 0$, $z \neq 0$ and $x \neq z$. In particular, then (5.6) holds for all $\varepsilon \leq z \leq \pi + \varepsilon$ after both ψ and $f_{1,\varepsilon}$ are replaced by f. However, then the right hand side of (5.6) is continuously differentiable, and we can conclude that f is continuously differentiable on $(0, 2\pi)$. This argument can then be iterated once more to conclude that f must be twice continuously differentiable on $(0, 2\pi)$ (this way even smoothness could be proved, but we will not need this property here).

We next prove that we can choose $f(0)$ so that f is continuous and $f'(x)$ has a limit for both $x \searrow 0$ and for $x \nearrow 2\pi$. Since $h(x, 2\pi - x) = \pi - x$ for all $x \in I$, we have for all $x \in (0, \pi)$,

$$f(x) - f(2\pi - x) + f(\pi - x) - f(\pi + x) = 0. \tag{5.8}$$

If f is antisymmetric, $f(\pi) = 0$ and (5.8) implies that for all $x \in (0, \pi)$, $f(x) = f(\pi + x)$. Therefore, in this case f is continuously differentiable at $x = 0$, after we define $f(2\pi n) = 0$, $n \in \mathbb{Z}$.

Assume then that f is symmetric which implies $f'(z) = -f'(2\pi - z)$. Let us consider values $0 < z < x < 2\pi$, when (5.3) holds. Differentiating (5.7) with respect to x and z yields

$$f'(x) + \partial_x h f'(h) - (1 + \partial_x h) f'(x - z + h) = 0, \tag{5.9}$$

$$-f'(z) + \partial_z h f'(h) - (-1 + \partial_z h) f'(x - z + h) = 0. \tag{5.10}$$

We multiply the second equality by $(1+\partial_x h)$, and then use the first one to eliminate $f'(x - z + h)$. This proves that

$$(1 - \partial_z h) f'(x) - (1 + \partial_x h) f'(z) + (\partial_x h + \partial_z h) f'(h) = 0. \tag{5.11}$$

We divide the equality by $1 - \partial_z h$ and consider taking the limit $x \to 2\pi$ for a fixed z. Then $h \to z - 2\pi$, and the partial derivatives converge as (see (5.17) to obtain explicit formulae from which these can be checked)

$$\partial_x h(x, z) \to -(1 + t^2) \quad \text{and} \quad \partial_z h(x, z) \to 1, \tag{5.12}$$

where $t = \tan \frac{2\pi - z}{4}$. Since $1 - \partial_z h \to 0$, we need to compute the limit more carefully. Let us fix for definiteness, $z = \pi$, when a straightforward computation shows that $\partial_z \partial_x h(x, z) \to 1$, and thus by L'Hospital's rule,

$$\lim_{x \nearrow 2\pi} \frac{f'(z) - f'(h(x, z))}{1 - \partial_z h(x, z)} = \lim_{x} \frac{f''(h(x, z)) \partial_x h(x, z)}{\partial_x \partial_z h(x, z)} = -2f''(\pi). \tag{5.13}$$

We can then use this result in (5.11) to prove that the limit of $f'(x)$ exists when $x \nearrow 2\pi$, and thus by symmetry the same is true about the limit when $x \searrow 0$. Since this implies that f' is bounded on $[0, 2\pi]$, we can also conclude that the limit $c = \lim_{x\to 0} f(x)$ exists, and we can make f continuous by defining $f(2\pi n) = c$ for all $n \in \mathbb{Z}$.

We can thus assume that f is continuous and periodic on $\mathbb{R}$, continuously differentiable on $(0, 2\pi)$, and that $a = \lim_{x\searrow 0} f'(x)$ and $b = \lim_{x\nearrow 2\pi} f'(x)$ exist. We also have $b = -a$, if f is symmetric, and $b = a$, if f is antisymmetric. Let us now consider any $0 < z \le \pi$, and $x = 2\pi - \varepsilon$ for $0 < \varepsilon < 2\pi - z$. As proven earlier, in the limit $\varepsilon \searrow 0$, $x - z + h \searrow 0$, and we get from (5.7) that when $\varepsilon \to 0$,

$$\begin{aligned} \frac{f(2\pi + h) - f(z)}{2\pi - x} &= \frac{f(x + h - z) - f(0) + f(2\pi) - f(x)}{2\pi - x} \\ &\to -(1 + \partial_x h)a + b. \end{aligned} \tag{5.14}$$

The left hand side converges to $-f'(z)\partial_x h$, and, since by (5.12) $\partial_x h(2\pi, z) = -(1 + t^2) < 0$, we have proven that

$$f'(z) = a - \frac{1}{1 + t^2}(a - b), \tag{5.15}$$

where $\frac{1}{1+t^2} = \cos^2 \frac{2\pi - z}{4} = \frac{1}{2}\left(1 - \cos\frac{z}{2}\right) = \frac{1}{2} - \omega'(z)$. Therefore, for $0 < z \le \pi$ we need to have

$$f'(z) = \frac{a + b}{2} + (a - b)\omega'(z). \tag{5.16}$$

If f is antisymmetric, then $f'(z) = a$ for $0 < z \le \pi$. Since then also $f'(2\pi - z) = a$, we must have $f(z) = c + az$ for $0 < z < 2\pi$. However, as also $f(0) = 0 = f(2\pi)$, we need to have $c = 0 = a$, and thus the only antisymmetric solution is the trivial solution $f = 0$. If f is symmetric, $b = -a$, and $f'(2\pi - z) = -f'(z)$. Thus then $f'(z) = 2a\omega'(z)$, for $0 < z < 2\pi$, and there is $c \in \mathbb{C}$ such that $f(z) = c + 2a\omega(z)$ for all $z \in \mathbb{R}$. Therefore, f is in both cases a trivial collisional invariant, and since $\psi = f$ almost everywhere, we have arrived at the conclusion made in the Theorem.

We still need to prove (5.5). Using the shorthand notations $t = \tan\frac{x-z}{4}$, $c = \cos\frac{x+z}{4}$, we can write

$$\begin{aligned} 2\partial_x \Phi(x, z) + 1 &= \frac{c(1 + t^2) - t\sqrt{1 - c^2}}{\sqrt{1 - c^2 t^2}} + 1 \\ &= \frac{1 + t^2}{\sqrt{1 - c^2 t^2}}\left(c + \frac{1 - t^2}{1 + t^2}\frac{1}{t\sqrt{1 - c^2} + \sqrt{1 - c^2 t^2}}\right). \end{aligned} \tag{5.17}$$

For ε, x, z as above, i.e., for $0 < \varepsilon < \frac{\pi}{4}$, $\varepsilon \le z \le \pi + \varepsilon$, and $2\pi - \varepsilon \le x \le 2\pi$,

$$\frac{\pi}{4} - \frac{\varepsilon}{2} \le \frac{x - z}{4} \le \frac{\pi}{2} - \frac{\varepsilon}{4} \quad \text{and} \quad \frac{\pi}{2} + \frac{z - \varepsilon}{4} \le \frac{x + z}{4} \le \frac{3\pi}{4} + \frac{\varepsilon}{4}. \tag{5.18}$$

Thus $t \geq \tan\left(\frac{\pi}{4} - \frac{\varepsilon}{2}\right) > 0$, and with $c' = \cos\frac{\pi-\varepsilon}{4}$

$$-1 < -c' \leq c \leq \cos\left(\frac{\pi}{2} + \frac{z-\varepsilon}{4}\right) = -\sin\frac{z-\varepsilon}{4} \leq 0. \tag{5.19}$$

If also $z \leq \frac{\pi}{2} - \varepsilon$, then $x - z \geq \frac{3\pi}{2}$ and thus $t \geq \tan(3\pi/8) > 2$. In this case, we can estimate the first term using $c \leq 0$, which yields

$$2\partial_x \Phi(x,z) + 1 \leq -\frac{t^2-1}{t+1} = -(t-1) \leq -1. \tag{5.20}$$

Otherwise, $z - \varepsilon > \frac{\pi}{2} - 2\varepsilon \geq \frac{\pi}{4}$. If $t \geq 1$, we find

$$2\partial_x \Phi(x,z) + 1 \leq c \leq -\sin\frac{z-\varepsilon}{4} \leq -\sin\frac{\pi}{16} < 0. \tag{5.21}$$

On the other hand, if $t < 1$, then by

$$\frac{1-t^2}{1+t^2} = \cos\frac{x-z}{2} \leq \cos\left(\frac{\pi}{2} - \varepsilon\right) = \sin\varepsilon, \tag{5.22}$$

we have

$$\begin{aligned} 2\partial_x \Phi(x,z) + 1 &\leq \frac{1+t^2}{\sqrt{1-c^2t^2}}\Big(c + \frac{1-t^2}{1+t^2}\frac{1}{\sqrt{1-(c')^2}}\Big) \\ &\leq \frac{1+t^2}{\sqrt{1-c^2t^2}}\Big(-\sin\frac{\pi}{16} + \frac{\sin\varepsilon}{\sin\frac{\pi-\varepsilon}{4}}\Big). \end{aligned} \tag{5.23}$$

Since the term in the parenthesis approaches $-\sin\frac{\pi}{16} < 0$, when $\varepsilon \to 0$, there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the right hand side is less than $-\frac{1}{2}\sin\frac{\pi}{16}$. Thus we can conclude that for all such ε Equation (5.5) holds at least with $C = \frac{1}{2}\sin\frac{\pi}{16}$. This completes the proof of the Theorem.

6 Resolvent expansion (proof of Theorem 2.5)

Let us begin with the following corollary of the results proven in the previous sections.

Corollary 6.1 *The eigenspace of B with eigenvalue 1 is two-dimensional, and it is spanned by $V(x)^{1/2}$ and $\omega(x)V(x)^{1/2}$. Every $\psi \in L^2(I)$, such that $P\psi = -\psi$, is orthogonal to this eigenspace.*

Proof: Suppose ψ belongs to the above eigenspace of B. By Proposition 2.4, then $\tilde{\psi} = V^{-1/2}\psi = \omega W^{-1/2}\psi$ is a collisional invariant. Thus by Theorem 2.2, there are $c_1, c_2 \in \mathbb{C}$ such that $\omega W^{-1/2}\psi = c_1\omega + c_2$. By Lemma 4.1, both of the vectors $W^{1/2} = \omega V^{1/2}$ and $\omega^{-1}W^{1/2} = V^{1/2}$ belong to L^2 and are symmetric under P. This proves the results stated in the Corollary. □

To estimate error terms, we will rely on the following estimates:

Lemma 6.2 *There is $C > 0$ such that for any $0 < \lambda < 1$,*

$$\int_0^{2\pi} \mathrm{d}x \, \frac{\omega(x)}{W(x)+\lambda} \left(\sin \frac{x}{2}\right)^{-\frac{1}{2}} \leq C\lambda^{-\frac{1}{10}}. \tag{6.1}$$

Proof: Let $0 < \lambda < 1$ be arbitrary. By symmetry, the integrals over $[0, \pi]$ and $[\pi, 2\pi]$ are equal. On the other hand, by Lemma 4.1, we have

$$\int_0^{\pi} \mathrm{d}x \, \frac{\omega(x)}{W(x)+\lambda} \left(\sin \frac{x}{2}\right)^{-\frac{1}{2}} \leq \frac{1}{1+c_1} \int_0^{\pi} \mathrm{d}x \, \frac{s_x^{1/2}}{s_x^{5/3}+\lambda}. \tag{6.2}$$

The integral over $x \in [\frac{1}{2}\pi, \pi]$ is clearly bounded uniformly in λ. To estimate the integral over $[0, \frac{1}{2}\pi]$, we change the integration variable to $s = \lambda^{-3/5} s_x$, which shows that

$$\int_0^{\pi/2} \mathrm{d}x \, \frac{s_x^{1/2}}{s_x^{5/3}+\lambda} \leq 2\sqrt{2}\lambda^{\frac{3}{5}+\frac{3}{10}-1} \int_0^{\infty} \mathrm{d}s \, \frac{s^{1/2}}{s^{5/3}+1} \leq c\lambda^{-\frac{1}{10}}, \tag{6.3}$$

since the integral over s is finite. Thus (6.1) holds for some finite C. □

Lemma 6.3 *For $0 < \lambda < 1$, let*

$$\varphi_\lambda = B \frac{W^{\frac{1}{2}}}{W+\lambda} \omega'. \tag{6.4}$$

For any $0 < \varepsilon < \frac{3}{10}$ there is a constant $c_\varepsilon > 0$ such that for all $x \in I$ and λ,

$$|\varphi_\lambda(x)| \leq c_\varepsilon \lambda^{-\frac{1}{10}-\varepsilon} \left(V(x) \sin \frac{x}{2}\right)^{-\frac{1}{2}}. \tag{6.5}$$

Proof: Let $0 < \varepsilon < \frac{3}{10}$ be arbitrary. Applying the definitions of B and W, as well as the bound $|\omega'| \leq \frac{1}{2}$, we find

$$|\varphi_\lambda(x)| \leq \frac{1}{2} V(x)^{-1/2} \int_0^{2\pi} \mathrm{d}y \, \left(|K_1(x,y)| + 2|K_2(x,y)|\right) \frac{\omega(y)}{W(y)+\lambda}. \tag{6.6}$$

By Lemma 6.2 and Eq. (3.18), the second term in the sum satisfies

$$\int_0^{2\pi} \mathrm{d}y \, 2|K_2(x,y)| \frac{\omega(y)}{W(y)+\lambda} \leq C \left(\sin \frac{x}{2}\right)^{-\frac{1}{2}} \lambda^{-\frac{1}{10}}, \tag{6.7}$$

and thus leads to a bound of the desired form.

To the first term we apply Lemma 4.2, which shows that

$$\int_0^{2\pi} \mathrm{d}y\, |K_1(x,y)| \frac{\omega(y)}{W(y)+\lambda} \le C \left|\sin\frac{x}{2}\right|^{-\frac{1}{2}} \\ \times \Big[\int_0^{y_1(x)} \mathrm{d}y \frac{1}{\sqrt{y_1(x)-y}} \frac{\omega(y)}{W(y)+\lambda} + \int_{y_2(x)}^{2\pi} \mathrm{d}y \frac{1}{\sqrt{y-y_2(x)}} \frac{\omega(y)}{W(y)+\lambda}\Big]. \tag{6.8}$$

Changing the integration variable to $y' = 2\pi - y$ in the second integral reveals that it is equal to the first integral, if $y_1(x)$ is replaced by $2\pi - y_2(x)$. Thus it is sufficient to inspect the first integral. We estimate it using Lemma 4.1 and Hölder's inequality for $p' = \frac{6}{3+10\varepsilon} < 2$, $q' = \frac{6}{3-10\varepsilon} > 2$. This shows that for any $y_1 \in I$,

$$\int_0^{y_1} \mathrm{d}y \frac{1}{\sqrt{y_1-y}} \frac{\omega(y)}{W(y)+\lambda} \\ \le \frac{1}{1+c_1} \left[\int_0^{y_1} \mathrm{d}y (y_1-y)^{-\frac{p'}{2}}\right]^{\frac{1}{p'}} \left[\int_0^{2\pi} \mathrm{d}y \left(\frac{\omega(y)}{\omega(y)^{5/3}+\lambda}\right)^{q'}\right]^{\frac{1}{q'}}. \tag{6.9}$$

The first factor is an ε-dependent, finite constant, and the second factor can be estimated as in the proof of Lemma 6.2: first we use symmetry to reduce the estimate to $[0,\pi]$, then $\omega(y) > 0$ on $[\pi/2,\pi]$ to bound the integral on this interval by a constant, and finally on the interval $[0,\pi/2]$ we change the integration variable to $s = \lambda^{-3/5} \sin\frac{y}{2}$. This shows that there is a constant c such that

$$\int_0^{2\pi} \mathrm{d}y \left(\frac{\omega(y)}{\omega(y)^{5/3}+\lambda}\right)^{q'} \le c\lambda^{\frac{3}{5}+q'(\frac{3}{5}-1)} \int_0^{\infty} \mathrm{d}s \left(\frac{s}{s^{5/3}+1}\right)^{q'} \le c'\lambda^{-\frac{2}{5}q'+\frac{3}{5}}, \tag{6.10}$$

where the integral over s is finite, since $q' > 2 > \frac{3}{2}$. Therefore, as $-\frac{2}{5} + \frac{3}{5q'} = -\frac{1}{10} - \varepsilon$, (6.9) implies that (6.5) holds also for the first term in the sum. □

This has the following immediate corollary:

Corollary 6.4 *For any $0 < \varepsilon < \frac{3}{5}$ there is a constant $c_\varepsilon > 0$ such that $\|\varphi_\lambda\|^2 \le c_\varepsilon \lambda^{-\frac{1}{5}-\varepsilon}$ for all $0 < \lambda < 1$. In addition, $P\varphi_\lambda = -\varphi_\lambda$.*

Proof: Let $\varepsilon' = \varepsilon/2$. Then by Lemma 6.3,

$$\|\varphi_\lambda\|^2 = \int_0^{2\pi} \mathrm{d}x\, |\varphi_\lambda(x)|^2 \le c'_{\varepsilon'} \lambda^{-\frac{1}{5}-2\varepsilon'} \int_0^{2\pi} \mathrm{d}x \left(V(x)\sin\frac{x}{2}\right)^{-1}. \tag{6.11}$$

By Lemma 4.1, there is C such that $V(x)\sin(x/2) \ge C(\sin(x/2))^{2/3}$, and thus the remaining integral over x is finite. This implies that there is c_ε such that $\|\varphi_\lambda\|^2 \le c_\varepsilon \lambda^{-\frac{1}{5}-\varepsilon}$. Since both B and W commute with P, and ω' is antisymmetric, it follows that φ_λ is antisymmetric. □

Armed with the above results, we can prove the main theorem. We claim that for any $0 < \varepsilon < \frac{\alpha}{2}$,

$$R(\lambda) = \left\langle \omega', \frac{1}{\lambda + W}\omega' \right\rangle + \mathcal{O}(\lambda^{-\frac{\alpha}{2}-\varepsilon}). \tag{6.12}$$

This implies that only the first term of a resolvent expansion needs to be considered for the limit (2.13). However, then

$$\left\langle \omega', \frac{\lambda^\alpha}{\lambda + W}\omega' \right\rangle = \frac{1}{4}\int_0^{2\pi} \mathrm{d}x \, \frac{\lambda^{\frac{2}{5}}}{\lambda + W(x)} \cos^2 \frac{x}{2}, \tag{6.13}$$

and we can use the symmetry of the integrand to reduce the integration region to $[0, \pi]$, while gaining a factor of 2. We then change variables to $s = \lambda^{-3/5} \sin \frac{x}{2}$. By Lemma 4.1, the remaining integrand is dominated by $\frac{1}{1+c_1 s^{5/3}}$ which is integrable on $(0, \infty)$. Thus

$$\lim_{\lambda \to 0^+} \left\langle \omega', \frac{\lambda^\alpha}{\lambda + W}\omega' \right\rangle = c_0 = \int_0^\infty \mathrm{d}s \, \frac{1}{1 + w_0 s^{\frac{5}{3}}}. \tag{6.14}$$

Clearly, $0 < c_0 < \infty$, as claimed in the Theorem.

To estimate the "error term" in (6.12) assume $0 < \lambda < 1$ is given. Let $A = W^{\frac{1}{2}} B W^{\frac{1}{2}}$, when A is a bounded operator and $\tilde{L} = W - A$. We use the resolvent expansion of $\tilde{L}$ up to the second order,

$$\frac{1}{\lambda + \tilde{L}} = \frac{1}{\lambda + W} + \frac{1}{\lambda + W} A \frac{1}{\lambda + W} + \frac{1}{\lambda + W} A \frac{1}{\lambda + \tilde{L}} A \frac{1}{\lambda + W} \tag{6.15}$$

where, since $W \geq 0$, $(\lambda + W)^{-1}$ is a bounded, positive operator. Therefore, denoting $\phi_\lambda = A \frac{1}{\lambda + W}\omega'$,

$$R(\lambda) = \left\langle \omega', \frac{1}{\lambda + W}\omega' \right\rangle + \left\langle \omega', \frac{1}{\lambda + W} A \frac{1}{\lambda + W}\omega' \right\rangle + \left\langle \phi_\lambda, \frac{1}{\lambda + \tilde{L}}\phi_\lambda \right\rangle. \tag{6.16}$$

Using Proposition 2.4, Corollary 6.1, and Lemmas 6.2–6.4, we can now prove (6.12). To estimate the first correction we use Lemmas 6.2 and 6.3:

$$\begin{aligned} \left| \left\langle \omega', \frac{1}{\lambda + W} A \frac{1}{\lambda + W}\omega' \right\rangle \right| &= \left| \left\langle \frac{W^{\frac{1}{2}}}{W + \lambda}\omega', \varphi_\lambda \right\rangle \right| \\ &\leq \int_0^{2\pi} \mathrm{d}x \, \frac{\omega(x)}{W(x) + \lambda} V(x)^{\frac{1}{2}} |\varphi_\lambda(x)| \leq c_\varepsilon C \lambda^{-\frac{1}{5}-\varepsilon}. \end{aligned} \tag{6.17}$$

This proves that the first correction is of the claimed order, and we only need to inspect the final term in (6.16).

Firstly, $\phi_\lambda = W^{\frac{1}{2}}\varphi_\lambda$, and thus

$$\left\langle \phi_\lambda, \frac{1}{\lambda + \tilde{L}}\phi_\lambda \right\rangle = \left\langle \varphi_\lambda, \frac{1}{1 - B + \lambda W^{-1}}\varphi_\lambda \right\rangle. \tag{6.18}$$

Here, by Lemma 4.1, we have the operator inequalities

$$1 - B + \lambda W^{-1} \geq 1 - B + \frac{\lambda}{c_2} \geq \frac{\lambda}{c_2} > 0. \tag{6.19}$$

where $\frac{\lambda}{c_2}$ is proportional to the unit operator, and thus commutes with B. Therefore,

$$(1 - B + \lambda W^{-1})^{-1} \leq (1 - B + \frac{\lambda}{c_2})^{-1} \tag{6.20}$$

implying

$$0 \leq \left\langle \phi_\lambda, \frac{1}{\lambda + \tilde{L}} \phi_\lambda \right\rangle \leq \left\langle \varphi_\lambda, \frac{1}{1 + \lambda/c_2 - B} \varphi_\lambda \right\rangle. \tag{6.21}$$

By Proposition 2.4, B is compact, and its spectrum consists of isolated eigenvalues (apart from zero). Since also $B \leq 1$, we can order the eigenvalues so that $1 = \lambda_1 > \lambda_2 > \ldots$. In particular, then $\delta = 1 - \lambda_2 > 0$. Since φ_λ is antisymmetric, it is orthogonal to the eigenspace of B with eigenvalue 1 by Corollary 6.1. Therefore, using the spectral decomposition of B, we find that

$$\left\langle \varphi_\lambda, \frac{1}{1 + \lambda/c_2 - B} \varphi_\lambda \right\rangle \leq \frac{1}{\delta + \lambda/c_2} \|\varphi_\lambda\|^2 \leq \frac{c}{\delta} \lambda^{-\frac{1}{5} - \varepsilon}, \tag{6.22}$$

where we used the estimate in Corollary 6.4. Then we can conclude from (6.21) that (6.12) holds. This completes the proof of Theorem 2.5.

A Integral operators

Given a positive measure μ on X, any function $K : X \times X \to \mathbb{C}$, which is measurable in $\mu \times \mu$, can be used to define an operator T in $L^2(\mu)$ by the formula

$$(Tf)(x) = \int_X \mu(\mathrm{d}y) K(x, y) f(y). \tag{A.1}$$

More precisely, we define T as a possibly unbounded operator with the domain

$$D(T) = \left\{ f \in L^2(\mu) \,\middle|\, \int \mu(\mathrm{d}x) \left(\int \mu(\mathrm{d}y) |K(x,y)|\, |f(y)| \right)^2 < \infty \right\}. \tag{A.2}$$

K is then called the integral kernel of the integral operator T.

We need here only the following convenient estimate for an operator norm of such integral operators.

Proposition A.1 *Let μ be a positive measure on X, and assume that $A : X \times X \to \mathbb{C}$ is measurable with respect to $\mu \times \mu$ and satisfies $A(x,y)^* = A(y,x)$ for almost every $(x,y) \in X^2$. Consider any measurable $\phi : X \to \mathbb{C}$, let $B(x,y) =$*

$\phi(x)^* A(x,y)\phi(y)$, and let T denote the corresponding integral operator. If there exists $\alpha \in \mathbb{R}$ such that

$$C_\alpha = \operatorname*{ess\,sup}_x \left(|\phi(x)|^{2-\alpha} \int_X \mu(\mathrm{d}y)\, |A(x,y)|\, |\phi(y)|^\alpha \right) < \infty, \tag{A.3}$$

then T is a bounded, self-adjoint operator on $L^2(\mu)$, and $\|T\| \le C_\alpha$.

Proof: Let $\alpha \in \mathbb{R}$ be given, and let us denote $\alpha' = 2 - \alpha$. Then $|\phi(x)|^2 = |\phi(x)|^\alpha |\phi(x)|^{\alpha'}$. (We apply here the usual convention used in connection with positive measures, that $0 \cdot \infty = 0$. Let also $0^0 = 1$). Thus for any $f \in L^2$, we have by the Schwarz inequality and Fubini's theorem

$$\begin{aligned}
&\int_{X^3} (\mu\times\mu\times\mu)(\mathrm{d}(x,y,z)) |f(x)|\, |B(z,x)|\, |B(z,y)|\, |f(y)| \\
&= \int_{X^3} (\mu\times\mu\times\mu)(\mathrm{d}(x,y,z)) \\
&\quad \times |f(x)| \left(|A(z,x)|\, |A(z,y)|\, |\phi(x)|^{\alpha'} |\phi(z)|^{\alpha+\alpha'} |\phi(y)|^\alpha \right)^{\frac12} \\
&\quad \times |f(y)| \left(|A(z,x)|\, |A(z,y)|\, |\phi(y)|^{\alpha'} |\phi(z)|^{\alpha+\alpha'} |\phi(x)|^\alpha \right)^{\frac12} \\
&\le \int \mu(\mathrm{d}x) |f(x)|^2 \Big(|\phi(x)|^{\alpha'} \int \mu(\mathrm{d}z) |A(x,z)|\, |\phi(z)|^\alpha \\
&\quad \times \Big[|\phi(z)|^{\alpha'} \int \mu(\mathrm{d}y) |A(z,y)|\, |\phi(y)|^\alpha \Big] \Big) \\
&\le C_\alpha^2 \|f\|^2 < \infty,
\end{aligned} \tag{A.4}$$

where we have used the symmetry of A. Therefore, $D(T) = L^2$, and since the left hand side of (A.4) is an upper bound for $\|Tf\|^2$, we have proven that $\|T\| \le C_\alpha$. As $B(x,y)^* = B(y,x)$ almost everywhere, T is then also self-adjoint. $\square$

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